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EDITORIAL NOTE

DIVERSE REASONS lie behind the founding of the *International Economic Review*. Three major motivating factors are (1) to provide a new truly international forum, (2) to facilitate, through such an international medium, the introduction of new national schools of thought (e.g., the newly developing Japanese school) to the world community of economists, and (3) to foster the development of quantitative economics. With the generous backing of the Kansai Economic Federation and the Rockefeller Foundation we have found it possible to bring our plans to a full realization. The Wharton School of Finance and Commerce of the University of Pennsylvania and the Institute of Social and Economic Research of Osaka University have happily joined in the spirit of international academic interchange to serve as institutional sponsors of this effort. These Universities have provided many facilities and given the needed encouragement to publish this new journal.

Our venture is purely scholarly. The editors and their staff reserve the exclusive right to determine the nature of the contents—to solicit, accept, or reject whatever articles or notes they deem to be acceptable for a journal of this type. The supporting and sponsoring bodies have given us the fullest academic freedom in this venture.

We have been pleased at the general acceptance of our proposal. An internationally distributed board of associate editors, composed of distinguished leaders in economics throughout the world, have agreed to serve in the work of the *Review*. Many have already given generously of their time in refereeing articles. Other referees—experts in particular fields of inquiry—have also helped us to review specialized articles. All areas of the world are not now represented on our board of associate editors, but we should be pleased to be able to extend the geographical scope of this board in the future.

American, European, and Japanese economists have already submitted works for publication. This aspect of response has been gratifying to us. We welcome more contributions, as works of merit are developed.

A strict and rigorous refereeing process is applied to each paper submitted. We aim to be impartial, critical, fair, and as speedy as possible. We feel that the contributions contained in this first issue exemplify the high standards that we have set.

In addition to being internationally minded, we maintain a strong interest in quantitative economics. We shall not refuse to give serious consideration to any competent work in non-quantitative economics, but expect to keep the general tone of this journal close to the area of quantitative economics. In our broad interpretation of this term we include mathematical economics, applied econometrics, econometric theory, and statistical economics. Economic theory that lays the foundation for quantitative studies is also within the scope of our interest.

At the outset, our journal will consist entirely of original articles of normal length. Eventually, we may develop a book review section if the demand for that facility is great enough. At present we are happy to consider review articles. We shall also be willing to publish shorter notes and comments that fall within our area of interest. Expository or survey articles on newly developing subjects in economics are also welcome.

While our guiding criterion is *quality* of material without particular attention to author's status, we do hope to encourage younger members of the world community of economists to submit their works to us.

EDITOR and CO-EDITOR

INTERNATIONAL ECONOMIC REVIEW
Vol. 1, No. 1, January, 1960

INVESTMENT DEMAND: AN EMPIRICAL CONTRIBUTION TO THE AGGREGATION PROBLEM*

BY J. C. G. BOOT AND G. M. DE WIT[1]

1. INTRODUCTION AND SUMMARY

PREVIOUS ANALYSES of problems arising in the study of aggregation have been largely theoretical in character. A notable exception is Balderston's and Whitin's article [1] on aggregation in input-output models, in which are shown the kind of divergencies to which various patterns of aggregation may lead. It is hardly necessary to stress the need for empirical analysis in this field, but there are particular difficulties to be solved, as the present paper will show. It is concerned not with input-output relations, but with relationships of the more "conventional" regression type.

The organization of this article is as follows. In Section 2 we give a brief re-formulation of Professor Theil's aggregation theory, which is applied in Section 4 to the coefficients of a macroeconomic investment equation. The underlying investment theory is due to Dr. Y. Grunfeld and is explained in Section 3. A considerable problem in the empirical analysis of aggregation is the fact that micro and macroparameters are unknown. At the micro-level this is solved by proceeding under the assumption that the parameters coincide with their least squares estimates; at the macro-level, by a statistical extension of aggregation theory such that the estimated macrocoefficients are formulated as the sum of a "true value," an aggregation bias, and a sampling error. For the particular case considered here, which deals with investment of ten large American corporations in the period 1935–1954, the aggregation bias of the multiplicative coefficients appears to be relatively small. The sampling errors can be attributed in a simple manner to individual firms and to separate sample years; the large firms appear to dominate these errors. The numerical characteristics of the auxiliary regression equations are also discussed, since the coefficients of these equations play a crucial role in the determination of the macroparameters (Section 5).

* Manuscript received June 8, 1959.

[1] During the preparation of this report Professor Theil has greatly assisted the authors by his highly stimulating remarks and valuable suggestions. Our thanks are also due to Dr. Grunfeld, cf. footnote 5.

The last two sections are devoted to the nature of the macrodisturbances. These are of great importance for the estimation of macroregressions because of the conditions which are usually imposed on them. Section 6 is concerned with the individual least-squares estimated macrodisturbances; it is shown that they are each the sum of a "true value," an aggregation bias, and a sampling error (just as are the macrocoefficients). In Section 7 the sample variance of the estimated macrodisturbances is analyzed, together with the first-order autocovariance. It is shown that a statistical appraisal of the significance of various components of the variance and the autocovariance is a useful tool in the analysis of macroregressions.

2. AGGREGATION BIAS OF MACROPARAMETERS

Our starting point is Theil's aggregation theory [5], which can be briefly summarized as follows. We assume that there are certain micro-units (e.g. firms or family households), indicated by the index $i = 1, \cdots, I$, which are aggregated to one single macro-unit. For each micro-unit we have a theory which states that some variable y_i depends linearly on, let us say, two other variables, x_{1i} and x_{2i}, except for a disturbance u_i. For example, a household's demand for dwelling-space (y_i) depends on its income (x_{1i}) and the number of family members (x_{2i}). If we add a t in parentheses to indicate the period to which the variables refer, we can write such a microtheory in the form:

$$y_i(t) = \alpha_i + \beta_{1i}x_{1i}(t) + \beta_{2i}x_{2i}(t) + u_i(t) \quad (i = 1, \cdots, I)\,, \tag{2.1}$$

where the α_i, β_{1i}, β_{2i} are microparameters. In general, these parameters will be different for different i, implying that the reaction of micro-unit i to a unit change in x_{1i} differs from the reaction of micro-unit j to a unit change in x_{1j} and similarly for x_{2i} and x_{2j}. The u's are regarded as random with zero expectation, while the x-variables are supposed to take nonstochastic values.[2]

We assume further that the microvariables y_i, x_{1i}, x_{2i} are aggregated by summation:

$$y(t) = \sum_{i=1}^{I} y_i(t)\,; \quad x_1(t) = \sum_{i=1}^{I} x_{1i}(t)\,; \quad x_2(t) = \sum_{i=1}^{I} x_{2i}(t)\,; \tag{2.2}$$

and that, on the analogy of (2.1), a linear relationship in these aggregates is introduced:

$$y(t) = \alpha + \beta_1 x_1(t) + \beta_2 x_2(t) + u(t)\,. \tag{2.3}$$

[2] No assumption is made here about the correlations of the u's, either about correlations over time, or about correlations for pairs (u_i, u_j). In Section 7, however, it will be assumed that there are no lagged correlations among the microdisturbances.

This procedure is fully in accordance with the conventional way of aggregating. If an individual household's demand for dwelling-space is assumed to depend on this household's family income and the number of its family members, then it is usually postulated that total demand depends on total personal income and the size of the population.

It will be clear that certain difficulties are associated with this procedure, because summation of all microequations (2.1) for $i = 1, \cdots, I$ does not express the aggregate y in terms of the aggregate x_1 and x_2, but in terms of all individual microvariables x_{1i}, x_{2i}. This problem is solved as follows. We assume that observations $y(t), x_1(t), x_2(t)$ on the aggregates y, x_1, x_2 are available for a series of consecutive periods $t = 1, \cdots, T$. Then we consider the least-squares regression of y on x_1 and x_2 for this sample of observations, after which the macroparameters α, β_1, β_2 of (2.3) are defined as the expectations of the least-squares regression coefficients; the macrodisturbances $u(t)$ for $t = 1, \cdots, T$ are then implicitly defined.[3] Essentially, this approach implies that we determine the expectations of the macroeconomic least-squares coefficients, given the microtheory as postulated in (2.1).

The next problem is: What is the relationship between the macroparameters of (2.3), thus interpreted, and the microparameters of (2.1)? The answer (which will not be proved here because we need a generalized result in the next sections) appears to be as follows. First, we consider the so-called *auxiliary regression equations*, which are the least-squares regressions of each of the explanatory microvariables on the two explanatory macrovariables during the period $t = 1, \cdots, T$:

$$(2.4) \qquad \begin{cases} x_{1i}(t) = A_{1i} + B_{1.1i}x_1(t) + B_{2.1i}x_2(t) + V_{1i}(t) \\ x_{2i}(t) = A_{2i} + B_{1.2i}x_1(t) + B_{2.2i}x_2(t) + V_{2i}(t) \,, \end{cases}$$

the A's and B's being regression coefficients and the V's residuals.[4] No special economic meaning is to be attached to the relations (2.4); they are introduced simply because their coefficients facilitate the way in which the macroparameters are expressed in terms of microparameters. These expressions are as follows:

[3] A similar procedure is available when we use not least-squares but any other linear unbiased estimation method (see Theil [5, pp. 116 seq.]), but this is not pursued here. It will also be clear that the method expounded here can easily be generalized for larger numbers of explanatory macrovariables. We confine ourselves here to the case of two macro-x's in view of the empirical case to be discussed below.

[4] Note that the A's and B's as well as the V's are nonstochastic because of our assumption that all x-variables take nonstochastic values.

$$(2.5)\qquad \begin{cases} \alpha = \sum_i \alpha_i + \sum_i A_{1i}\beta_{1i} + \sum_i A_{2i}\beta_{2i} \\ \beta_1 = \sum_i B_{1.1i}\beta_{1i} + \sum_i B_{1.2i}\beta_{2i} \\ \beta_2 = \sum_i B_{2.1i}\beta_{1i} + \sum_i B_{2.2i}\beta_{2i} \,. \end{cases}$$

It follows from (2.5) that it is *not* true that the constant term of the macrorelation (α) depends only on the constant terms of the microrelations, nor that β_1 depends only on the β_{1i} and that β_2 depends only on the β_{2i}. In other words, for each of the three macroparameters we find that they depend on *corresponding* as well as on *non-corresponding* parameters. This is a rather negative result, but it is fortunately possible to be somewhat more specific with respect to the weights of the microparameters in (2.5) (i.e., the coefficients of the auxiliary regressions). It can be proved that

$$(2.6)\qquad \begin{cases} \sum_i A_{1i} = \sum_i A_{2i} = 0 \\ \sum_i B_{1.1i} = \sum_i B_{2.2i} = 1 \\ \sum_i B_{1.2i} = \sum_i B_{2.1i} = 0 \,. \end{cases}$$

Comparing (2.5) and (2.6), we find that the weights of the non-corresponding microparameters always add up to zero, while the corresponding microparameters for β_1 and β_2 have weights that add up to one. In view of this result, it is worth-while to re-write the last two equations of (2.5) as follows:

$$(2.7)\qquad \begin{cases} \beta_1 = \frac{1}{I}\sum_i \beta_{1i} + \sum_i \left(B_{1.1i} - \frac{1}{I}\right)\beta_{1i} + \sum_i B_{1.2i}\beta_{2i} \\ \beta_2 = \frac{1}{I}\sum_i \beta_{2i} + \sum_i B_{2.1i}\beta_{1i} + \sum_i \left(B_{2.2i} - \frac{1}{I}\right)\beta_{2i} \,. \end{cases}$$

Taking the first equation of (2.5), together with (2.7), we can conclude that the constant term α of the macrorelation is equal to the sum of the constant terms of all microrelations, and that the multiplicative macroparameters β_1 and β_2 are equal to the (unweighted) means of the corresponding microparameters, apart from certain deviations which are called the *aggregation biases* of these macroparameters. In the case of α this bias is

$$\sum_i A_{1i}\beta_{1i} + \sum_i A_{2i}\beta_{2i} \,,$$

the first term representing the aggregation bias due to the β_{1i}, the second that due to the β_{2i}. Hence, for α the aggregation bias is

wholly due to noncorresponding microparameters. For β_1 the bias is

$$\sum_i \left(B_{1.i1} - \frac{1}{I}\right)\beta_{i1} + \sum_i B_{1.i2}\beta_{i2} ,$$

of which the first term represents the aggregation bias due to the corresponding microparameters β_{i1}. For β_2 the results are analogous.

3. GRUNFELD'S INVESTMENT THEORY[5]

The procedure described in the preceding section will be applied to $I = 10$ large American corporations, for which gross investment (including maintenance and repairs) is taken as dependent microvariable. A list of these firms, arranged in order of decreasing average investment during the period 1935–1954, is presented in Table 1; this table also contains the amounts of average investment and the abbreviations of the names of these firms that will be used in later tables. The aggregate amount of investment of this group of firms was about 7 per cent of total private investment in the United States in 1952.

Grunfeld rejects (observed) profits as an explanatory variable for investment; instead, he prefers expected profits, for which he takes as a measurable substitute the "market value of the firm," viz., the total value of outstanding stock at end-of-year stock market quotations[6]. He then introduces the "desired capital stock" at time t, $C^*(t)$, which is supposed to be a linear (increasing) function of the market value at t, $F(t)$:

$$C^*(t) = c_1 + c_2 F(t) . \tag{3.1}$$

Let $C(t)$ be the existing stock, then $c_1 + c_2F(t) - C(t)$ is desired net

[5] This is the correct place to describe our indebtedness to Dr. Y. Grunfeld and also to Professor Z. Griliches of the University of Chicago. Although it will be clear that we consider Grunfeld's theory and data [3] primarily as a convenient starting point of our aggregation analysis, it will be equally clear that we cannot claim credit for any conceivable merit which his theory may have, and that we are open to discredit for having followed him if his theory proves to have defects. We should also mention that Grunfeld and Griliches wrote a joint paper [4] on the aggregation side of Grunfeld's investment analysis as well as of Griliches' analysis of the demand for fertilizer, which was published in another paper [2]. The main difference between the present aggregation analysis and that of Grunfeld and Griliches is that the latter is confined to multiple correlation coefficients of micro and macroregressions. The present paper deals with micro and macroparameters as well as residual variances (and hence implicitly with multiple correlations). Another difference is that this paper includes some large corporations which were considered by Grunfeld only after the bulk of his computations was completed, and which were therefore excluded from his aggregation analysis.

[6] For the firms 2, 4, 6 and 8 of Table 1 the value was measured by Grunfeld at average December 31 and January 31 stock market quotations, which gave somewhat better results.

TABLE 1

LIST OF AMERICAN CORPORATIONS ANALYZED

No.	Firm	Abbreviation	Average amount of investment*
1	General Motors Corporation	G.M.	608.0
2	United States Steel Corporation	U.S. Steel	410.5
3	General Electric	G. Elec.	102.3
4	Chrysler Corporation	Chrys.	86.1
5	Atlantic Refining	Atl. Ref.	61.8
6	International Business Machines	I.B.M.	55.4
7	Union Oil	U. Oil	47.6
8	Westinghouse	West.	42.9
9	Goodyear Tire and Rubber	G. Year	41.9
10	Diamond Match	D. Match	3.1

* Measured in millions of dollars of 1947 purchasing power per year.

investment. Assuming that a constant fraction q_1 of desired net investment is carried out in the year between t and $t+1$, we get for actual net investment of that year

$$q_1\{C^*(t) - C(t)\} = q_1c_1 + q_1c_2F(t) - q_1C(t)\,. \tag{3.2}$$

Remembering that it is gross investment including maintenance and repairs that we want to describe, and assuming that replacement investment plus maintenance and repairs equals a constant fraction q_2 of the existing capital stock $C(t)$, we obtain

$$\begin{aligned} I(t+1) &= q_1\{C^*(t) - C(t)\} + q_2C(t) \\ &= q_1c_1 + q_1c_2F(t) + (q_2 - q_1)C(t)\,, \end{aligned} \tag{3.3}$$

where $I(t+1)$ is gross investment including maintenance and repairs in the year following time t. Empirical testing of this relation shows that it out-performs current profits as well as lagged profits; moreover, it offers predictions superior to those of conventional naive models in nearly all cases.[7] As for the numerical values of the coefficients, it seems reasonable to assume that the q's and c_2 are between 0 and 1; in addition, Grunfeld offers the suggestion that q_2 will usually be below 0.3. It will be noted that the coefficient of capital stock in (3.3) may be positive. This may seem strange for an investment equation, but it must be remembered that the dependent variable is not net investment but gross investment including maintenance and

[7] Either stating: "same investment as last year" [$I(t+1) = I(t)$], or "same change in investment as last year" [$\Delta I(t+1) = \Delta I(t)$, where Δ is the operator of first backward differences].

TABLE 2

MICROREGRESSIONS DESCRIBING INVESTMENT OF 10 AMERICAN CORPORATIONS*

	G.M.	U.S. Steel	G. Elec.	Chrys.	Atl. Ref.
$\bar{R}^2$	.907	.377	.653	.898	.624
Coeff. of F_{-1}	.119(.026)	.175(.074)	.027(.016)	.078(.020)	.162(.057)
Coeff. of C_{-1}	.371(.037)	.390(.045)	.152(.026)	.316(.029)	.003(.022)
Constant term	−149.78	−49.20	−9.96	−6.19	22.71
Est. var. disturbances	7537.15	8320.70	695.61	157.76	73.52
von Neumann ratio	1.09	1.09	1.08	2.02	2.39
	I.B.M.	U. Oil	West.	G. Year	D. Match
$\bar{R}^2$	.944	.722	.699	.606	.580
Coeff. of F_{-1}	.131(.031)	.088(.066)	.053(.016)	.075(.034)	.005(.027)
Coeff. of C_{-1}	.085(.100)	.123(.017)	.093(.056)	.082(.028)	.437(.080)
Constant term	−8.69	−4.50	−.48	−7.72	.16
Est. var. disturbances	58.45	79.34	93.55	74.07	1.05
von Neumann ratio	1.77	1.91	1.58	1.64	1.22

* Figures in parentheses are least-squares standard errors.

repairs.

The least-squares method applied to each of the 10 corporations gives rise to a number of estimates which are summarized in Table 2.[8] The squared coefficients of multiple correlation $\bar{R}^2$ (adjusted for loss of degrees of freedom) are in general satisfactorily high, though it should be remembered that a simple rising trend alone accounts for a rather large part of the variance of the dependent variable in most of the cases. The estimated coefficients are all positive, which is what one would reasonably hope for; some of the C-coefficients are above 0.3, which is rather large in terms of the discussion in the preceding paragraph. The standard errors of the coefficients are of the usual order of magnitude, but we should add that the large majority of the von Neumann ratios below the level 2 (indicating a positive first autocorrelation of the disturbances) suggests that these standard errors are on the optimistic side.[9]

The constant terms have only limited meaning, because capital stock has been measured as a deviation from the stock in 1933. There is also the obvious difficulty of the war years, which were characterized by much government intervention. We cannot exclude the

[8] The figures of the tables are generally specified in three or four decimal places. The actual computations have all been carried out in seven decimal places or more.

[9] The ratios are of the mean square successive difference to the variance of the least squares estimated microdisturbances.

possibility that this affects our results. It will, however, be clear that our interest is largely confined to the methodology of aggregation rather than the mechanism of investment decisions, so that we feel justified in neglecting this feature.

Comparing these results with the theoretical set-up of Section 2, we note that we now have a microtheory to which the aggregation procedure can be applied—viz., by interpreting y_t, x_{1t}, x_{2t} as I, F_{-1}, C_{-1} respectively for each of the corporations—except that we do not have a fully specified numerical microtheory. We do not have the microparameters $\alpha_t, \beta_{1t}, \beta_{2t}$; we have only estimates of these parameters. The only way to make any progress at all is to make certain assumptions about the values of the microparameters. We shall make the simplest assumption possible, viz. that the microparameters coincide with their least-squares estimates. Needless to say, this is a restrictive assumption and it may not be fulfilled. In the mimeographed version of this article we introduced an alternative assumption, which, however, showed generally small differences.

4. AGGREGATION BIAS AND IMPLIED SAMPLING ERRORS OF THE LEAST-SQUARES MACROCOEFFICIENTS

At the macro-level we have precisely the same parameter-estimate problem as at the micro-level; since we do not know the macroparameters, we can, at most, estimate them. A straightforward least-squares calculation gives for the macrorelation

$$I = -332.3 + \underset{(0.025)}{0.099}F_{-1} + \underset{(0.020)}{0.260}C_{-1}, \tag{4.1}$$

the von Neumann ratio of the estimated disturbances being 1.40.

This problem can, however, be solved in an elegant manner, without resort to additional assumptions. We shall do so in a general way, which can also be used for cases other than our investment problem. Let us write y for the column vector of values $y(t)$ taken by the dependent macrovariable (in the present case, the values of aggregate investment in each of the twenty years), and X for the matrix of values taken by the explanatory macrovariables. Here, X is of the order 20×3, the first column consisting of twenty elements which are all equal to 1 (corresponding with the constant term of the equation), while the second and third contain the values taken by the aggregate value of all firms and aggregate capital stock respectively. The vector of least-squares coefficients of the regression of y on X is then

$$\begin{bmatrix} a \\ b_1 \\ b_2 \end{bmatrix} = (X'X)^{-1}X'y \,, \tag{4.2}$$

where a, b_1, b_2 are the least-squares estimates of α, β_1, β_2 respectively. It will prove useful to introduce the matrix

$$\Xi = (X'X)^{-1}X' \,, \tag{4.3}$$

which is of order 3×20 in the present case. Let us write ξ_{0t} for a typical element of the first row of Ξ, and similarly ξ_{1t} and ξ_{2t} for the elements in the second and third row respectively. Then, comparing (4.2) and (4.3), we observe that

$$a = \sum_t \xi_{0t} y(t) \,; \quad b_1 = \sum_t \xi_{1t} y(t) \,; \quad b_2 = \sum_t \xi_{2t} y(t) \,. \tag{4.4}$$

This simple result has an important generalization, which we shall use frequently in what follows. Consider an arbitrary series of real numbers $z(1), \cdots, z(T)$ (T being 20 in our case); then

$$\sum_t \xi_{0t} z(t) \,, \quad \sum_t \xi_{1t} z(t) \,, \quad \sum_t \xi_{2t} z(t) \tag{4.5}$$

are the constant term and the two multiplicative coefficients of the least-squares regression of the z-variable on X. This follows immediately from the fact that the three expressions (4.5) are the components of $\Xi z = (X'X)^{-1}X'z$, z being the vector of values $z(t)$ taken by the z-variable.

Consider first the constant term, and combine (4.4), (2.2) and (2.1):

$$\begin{aligned} a &= \sum_t \xi_{0t} y(t) = \sum_t \xi_{0t} \sum_i y_i(t) \\ &= \sum_t \xi_{0t} \sum_i \{\alpha_i + \beta_{1i} x_{1i}(t) + \beta_{2i} x_{2i}(t) + u_i(t)\} \\ &= \sum_i \alpha_i \sum_t \xi_{0t} + \sum_i \beta_{1i} \sum_t \xi_{0t} x_{1i}(t) + \sum_i \beta_{2i} \sum_t \xi_{0t} x_{2i}(t) \\ &\qquad + \sum_t \xi_{0t} \sum_i u_i(t) \,. \end{aligned} \tag{4.6}$$

This expression is simplified as follows. First, we have $\sum \xi_{0t} = 1$;[10] second, $\sum \xi_{0t} x_{1i}(t)$ and $\sum \xi_{0t} x_{2i}(t)$ are the constant terms in the least-squares regressions of x_{1i} and x_{2i}, respectively, on the explanatory macrovariables x_1 and x_2. This follows from the remark made at the end of the preceding paragraph, and it implies that $\sum \xi_{0t} x_{1i}(t) = A_{1i}$,

[10] The proof is as follows. We can consider $\sum \xi_{0t}$ as the first component of the vector $\Xi\iota$, where ι is a column vector of unit elements of appropriate order. But ι is also the first column of X. Hence $\sum \xi_{0t}$ is the leading element of $\Xi X = (X'X)^{-1}X'X = I$, that is, it is equal to 1.

$\sum \xi_{0i} x_{2i}(t) = A_{2i}$, where A_{1i} and A_{2i} are the constant terms of the auxiliary regression (2.4). Hence we can write for (4.6)

$$\begin{aligned} a &= \sum_i \alpha_i + \sum_i A_{1i}\beta_{1i} + \sum_i A_{2i}\beta_{2i} + \sum_i \xi_{0i} \sum_i u_i(t) \\ &= \alpha + \sum_i \xi_{0i} \sum_i u_i(t) , \end{aligned} \tag{4.7}$$

α being given by (2.5). It is easily seen that α is the expectation of a, provided that the x-values (and hence the ξ's) are nonstochastic and that the microdisturbances have all zero expectation. We shall call $\sum \xi_{0i} \sum u_i(t)$ the *implied sampling error* of the constant-term estimate; this double sum can be interpreted as the constant term of the least-squares regression of the sum of the microdisturbances ($\sum_i u_i$) on the explanatory macrovariables x_1 and x_2.

The derivation of the multiplicative coefficients b_1 and b_2 is entirely similar. Let us take b_1 as defined in (4.4), and apply (2.2) and (2.1):

$$\begin{aligned} b_1 &= \sum_i \xi_{1i} y(t) = \sum_i \xi_{1i} \sum_i y_i(t) \\ &= \sum_i \alpha_i \sum_i \xi_{1i} + \sum_i \beta_{1i} \sum_i \xi_{1i} x_{1i}(t) + \sum_i \beta_{2i} \sum_i \xi_{1i} x_{2i}(t) \\ &\qquad + \sum_i \xi_{1i} \sum_i u_i(t) . \end{aligned}$$

It is easily seen that $\sum \xi_{1i} = 0$,[11] that $\sum \xi_{1i} x_{1i}(t) = B_{1.1i}$ and that $\sum \xi_{1i} x_{2i}(t) = B_{1.2i}$ as defined in (2.4), and that $\sum \xi_{1i} \sum u_i(t)$ is the coefficient of x_1 in the least-squares regression of the sum of the microdisturbances on the explanatory macrovariables x_1 and x_2. Hence

$$\begin{aligned} b_1 &= \sum_i B_{1.1i}\beta_{1i} + \sum_i B_{1.2i}\beta_{2i} + \sum_i \xi_{1i} \sum_i u_i(t) \\ &= \beta_1 + \sum_i \xi_{1i} \sum_i u_i(t) , \end{aligned} \tag{4.8}$$

β_1 (the expectation of b_1 under the conditions formulated in the preceding paragraph) being given by (2.5).[12] Similar results are obtained for b_2 by interchanging the indices 1 and 2.

Our analysis shows that the least-squares estimates a, b_1, b_2 are equal to their macroparameters as specified in (2.5), plus certain implied sampling errors. The macroparameters can be split up in terms of a "true value" and an aggregation bias. For the "true value" of α we take the sum of the contsant terms of the microequations ($\sum \alpha_i$),

[11] Proof: $\sum \xi_{1i}$ can be regarded as a nondiagonal element of the unit matrix of the preceding footnote.

[12] The sum-of-weight rules (2.6) are also easily verified. For example, $\sum A_{1i} = \sum \xi_{0i} x_{1i}(t)$ is the element in the first row and the second column of the unit matrix of the preceding footnotes, and hence equal to zero.

for that of β_1 and β_2 the average of the corresponding microparameters [$(1/I)\sum\beta_{1i}$ and $(1/I)\sum\beta_{2i}$ respectively]. The aggregation bias has already been discussed in Section 2.

Table 3 specifies these results for our investment problem numerically. It is seen that, as far as the multiplicative coefficients are concerned, the aggregation bias due to noncorresponding microparameters is of moderate size: for β_1 it is of the order of 10 per cent of the "true value," for β_2 only 1 per cent. This is an encouraging result, but it is of course rash to accept this as a general result. The bias due to corresponding microparameters is generally larger, in particular for β_2, but in our opinion this is less serious; we shall come back to this point at the end of Section 5. For the constant term the aggregation bias is much larger, and this is largely due to the fact that all of our variables take positive values, implying that the constant term measures the value of the dependent variable outside the range of observations. If the constant term were measured closer to the center of gravity of the observations, both the aggregation bias and the implied sampling error would be reduced.[13]

It will be observed that the implied sampling errors can be ascribed

TABLE 3

AGGREGATION BIAS AND IMPLIED SAMPLING ERRORS OF THE LEAST-SQUARES MACROCOEFFICIENTS

Constant Term		
Sum of corresponding microparameters		−213.6
Aggregation bias		−126.4
due to β_{1i}	59.3	
due to β_{2i}	−185.6	
Implied sampling error		7.7
Total: estimated coefficient		−332.3
Coefficient of F_{-1}		
Average of corresponding microparameters		.0913
Aggregation bias		.0147
due to corresponding microparameters	.0075	
due to noncorresponding microparameters	.0072	
Implied sampling error		−.0068
Total: estimated coefficient		.0993
Coefficient of C_{-1}		
Average of corresponding microparameters		.2053
Aggregation bias		.0311
due to corresponding microparameters	.0289	
due to noncorresponding microparameters	.0022	
Implied sampling error		.0238
Total: estimated coefficient		.2602

[13] Bias and sampling error vanish entirely when the constant term is measured at the center of gravity.

in a unique way to separate firms and years. For example, if we take the sampling error of the constant term, we find that it consists of terms $\xi_{0i}u_i(t)$, each of which can be allocated to some firm i in some year t. This allocation is shown in tabular form in the Appendix to this paper (Tables 11–13); here, we confine ourselves to the marginal totals, i.e., to

$$\sum_t \xi_{0i}u_i(t)\,,$$

which is the total contribution of firm i to the sampling error of the constant term, and to

$$\xi_{0i}\sum_i u_i(t)\,,$$

which is the total contribution of year t. A summary of these contributions—for the constant term as well as for the multiplicative coefficients—is given in Table 4. The results show that the implied sampling errors are the result of numerous positive and negative contributions, but also that some of these are much more dominant than others. For example, the contributions of the smallest firms are much smaller than those of the two largest (General Motors and U.S. Steel); an inspection of the tables in the Appendix shows that this also holds for most of the individual years. Further, it is interesting to observe that the years 1944–46, which mark the end of the war and the transitional period thereafter, contribute very little.

5. DISCUSSION OF THE AUXILIARY REGRESSION EQUATIONS

The analysis of the preceding section was to a large extent based on the coefficients of the auxiliary equations (2.4). It is worth-while to analyze these in somewhat more detail, first because very little is known about them at the empirical level, second because an analysis of these equations is not hampered by our lack of complete knowledge of the microtheory. We take all x-values as they are observed, which implies a complete knowledge of the auxiliary regression equations.

The coefficients $A_{1i}, A_{2i}, B_{1,ii}, \cdots$ of the auxiliary regressions are presented in Table 5. As such, these coefficients have relatively little meaning from an economic point of view, since the variables of the auxiliary equations are tied together in a way which is economically irrelevant. We could, however, perhaps say that in the "ideal" case the explanatory microvariables move proportionally to the corresponding aggregate. This case is indeed ideal in the sense that all macroparameters are then independent of noncorresponding microparameters. This follows directly from the fact that addition of the microequations,

TABLE 4

CONTRIBUTIONS OF SEPARATE FIRMS AND SEPARATE YEARS TO THE SAMPLING ERRORS OF THE ESTIMATED MACROPARAMETERS

Contributions of separate firms

Firm	Constant Term	Coeff. of F_{-1}	Coeff. of C_{-1}
G.M.	30.8	−.0039	.0040
U.S. Steel	7.9	−.0062	.0213
G. Elec.	−10.8	.0012	−.0006
Chrys.	10.0	−.0012	.0010
Atl. Ref.	−14.6	.0016	−.0010
I.B.M.	−6.0	.0006	−.0000
U. Oil	−11.0	.0011	−.0005
West.	−1.8	.0002	.0001
G. Year	2.0	.0000	−.0007
D. Match	1.2	−.0002	.0002
Total	7.7	−.0068	.0238

Contributions of separate years

Year	Constant Term	Coeff. of F_{-1}	Coeff. of C_{-1}
1935	5.2	−.0003	−.0009
36	−76.8	.0065	.0055
37	−11.9	.0006	.0034
38	−2.5	.0002	.0008
39	−34.6	.0041	−.0057
40	−53.3	.0060	−.0069
41	54.1	−.0058	.0057
42	38.2	−.0038	.0031
43	−36.2	.0057	−.0054
44	4.1	−.0010	.0012
45	0.5	−.0000	−.0001
46	−4.2	.0002	.0002
47	38.6	−.0031	.0004
48	−1.0	.0014	−.0029
49	11.5	−.0028	.0047
50	2.6	−.0051	.0123
51	−46 3	.0034	.0013
52	61.7	−.0089	.0100
53	−1.1	.0005	−.0013
54	59.1	−.0046	−.0016
Total	7.7	−.0068	0238

combined with an elimination of the explanatory microvariables by means of the proportionalities just-mentioned, gives a simple equation —without aggregation bias—in macrovariables only. Now this ideal case can be handled conveniently as follows. Consider the auxiliary equations (2.4) and average them over time, after which the results are divided by the time average of x_{1i} and x_{2i} respectively. Taking account of the fact that the residuals have zero mean, we obtain

$$1 = \frac{A_{1i}}{\bar{x}_{1i}} + B_{1.1i}\frac{\bar{x}_1}{\bar{x}_{1i}} + B_{2.1i}\frac{\bar{x}_2}{\bar{x}_{1i}} \tag{5.1}$$

$$1 = \frac{A_{2i}}{\bar{x}_{2i}} + B_{1.2i}\frac{\bar{x}_1}{\bar{x}_{2i}} + B_{2.2i}\frac{\bar{x}_2}{\bar{x}_{2i}}\,, \tag{5.2}$$

where $\bar{x}_{1i}, \bar{x}_{2i}, \bar{x}_1, \bar{x}_2$ are the means of x_{1i}, x_{2i}, x_1, x_2 respectively. It is easy to see that if all microvariables move proportionally to the corresponding aggregate, $B_{1.1i}$ must be equal to $\bar{x}_{1i}/\bar{x}_1$; in other words, the second term on the right of (5.1) must be unity. It is also easily verified that the first and the third term on the right must then vanish; similarly, the third term on the right of (5.2) must be unity, the two others being zero.

The coefficients $A_{1i}/\bar{x}_{1i}, A_{2i}/\bar{x}_{2i}, \cdots$ are also specified in Table 5. It appears that their average values over all 10 firms are close to the value of the ideal case, but that there are some sizable discrepancies for individual firms, leading to standard deviations of the order of 0.3 or 0.4. In none of the six cases do we find a significant difference between the computed average and the ideal value (significance being judged by dividing the difference by $1/\sqrt{10}$ times the standard deviation); but it must be remembered, of course, that such an appraisal of significance is hampered by the fact that all quantities entering the auxiliary equations are taken as nonstochastic.

It is also of some interest to observe that the "ideal case" enables us to formulate the "true value" of a multiplicative macroparameter in a slightly different manner. In Section 2 we took the unweighted mean for this purpose, but we may as well take a weighted mean, the weights being $\bar{x}_{1i}/\bar{x}_1$ for β_{1i} and $\bar{x}_{2i}/\bar{x}_2$ for β_{2i}. This means that the first equation of (2.7) is replaced by

$$\beta_1 = \sum_i \frac{\bar{x}_{1i}}{\bar{x}_1}\beta_{1i} + \sum_i \left(B_{1.1i} - \frac{\bar{x}_{1i}}{\bar{x}_1}\right)\beta_{1i} + \sum_i B_{1.2i}\beta_{2i}\,, \tag{5.3}$$

the second equation of (2.7) being changed similarly. It follows from (5.3) that the aggregation bias due to noncorresponding microparameters remains as it is, but the true value and the aggregation bias due

TABLE 5

COEFFICIENTS OF THE AUXILIARY EQUATIONS

Firm	A_{1t}	$B_{1.1t}$	$B_{2.1t}$	$\frac{A_{1t}}{\bar{x}_{1t}}$	$B_{1.1t}\frac{\bar{x}_1}{\bar{x}_{1t}}$	$B_{2.1t}\frac{\bar{x}_2}{\bar{x}_{1t}}$
G.M.	−486.65	.4615	−.0622	−.112	1.152	−.040
U.S. Steel	756.89	.1309	−.0729	.384	.718	−.102
G. Elec.	−93.36	.2077	−.0770	−.048	1.158	−.109
Chrys.	−38.06	.0720	−.0172	−.055	1.123	−.069
Atl. Ref.	20.58	.0107	.0345	.089	.500	.411
I.B.M.	−45.39	.0122	.1208	−.108	.314	.794
U. Oil	27.14	.0121	−.0029	.181	.873	−.054
West.	−236.19	.0671	.0657	−.352	1.082	.270
G. Year	38.04	.0236	.0146	.114	.765	.121
D. Match	57.00	.0021	−.0033	.804	.327	−.130
Average				.090	.801	.109
Standard deviation				.320	.333	.301

Firm	A_{2t}	$B_{1.2t}$	$B_{3.2t}$	$\frac{A_{2t}}{\bar{x}_{2t}}$	$B_{1.2t}\frac{\bar{x}_1}{\bar{x}_{2t}}$	$B_{3.2t}\frac{\bar{x}_2}{\bar{x}_{2t}}$
G.M.	−510.19	.0127	.3698	−.787	.213	1.574
U.S. Steel	15.15	.0047	.0828	.051	.173	.776
G. Elec.	80.96	−.0108	.1581	.202	−.293	1.090
Chrys.	−155.89	.0108	.0580	−1.286	.965	1.321
Atl. Ref.	231.69	−.0062	.1167	.476	−.138	.662
I.B.M.	−3.91	−.0004	.0409	−.037	−.046	1.084
U. Oil	157.59	−.0048	.0759	.500	−.165	.665
West.	18.44	−.0037	.0389	.215	−.469	1.253
G. Year	167.13	−.0025	.0572	.561	−.091	.530
D. Match	−.96	.0002	.0016	−.162	.440	.723
Average				−.027	.059	.968
Standard deviation				.594	.412	.346

to corresponding microparameters are affected. Applying this to β_1, we find that the true value and the "corresponding" bias, which originally were 0.0913 and 0.0075, become 0.1048 and −0.0060, respectively. We observe that the sign of the aggregation bias due to corresponding microparameters is changed if we interpret this bias in the weighted manner of (5.3). For β_3 the changes are relatively smaller: True value and "corresponding" bias change from 0.2053 and 0.0289 to 0.1953 and 0.0389 respectively.

6. THE LEAST-SQUARES ESTIMATED MACRODISTURBANCES: (i) THE INDIVIDUAL ERROR TERMS

So far we have confined our attention to the macroparameters α, β_1, β_2, and to their least-squares estimates. But it is also interesting to analyze the discrepancies between the observed dependent macrovalues and the corresponding values which are "predicted" by the macroequation. This leads us to an analysis of the estimated macrodisturbances, which is largely similar to the analysis of the estimated macrocoefficients.

Let us write $v(t)$ for the least-squares estimated macrodisturbance in t, and v for the vector of these estimates. Then

$$v = y - X(X'X)^{-1}X'y = My \,, \tag{6.1}$$

where M is a square matrix of order $T(=20$ in our case) with typical element $m_{tt'}$:

$$M = [m_{tt'}] = I - X(X'X)^{-1}X' \,, \tag{6.2}$$

I being the unit matrix of order T. It is important to note that M is analogous to Ξ in the following sense: if we postmultiply M by any column vector z of T real components, then the resulting column Mz can be interpreted as the residual vector of the least-squares regression of z on X.

Consider then an individual error term $v(t)$, and combine (6.1) with (2.2) and (2.1):

$$\begin{aligned} v(t) &= \sum_{t'} m_{tt'} y(t') = \sum_{t'} m_{tt'} \sum_i y_i(t') \\ &= \sum_i \alpha_i \sum_{t'} m_{tt'} + \sum_i \beta_{1i} \sum_{t'} m_{tt'} x_{1i}(t') + \sum_i \beta_{2i} \sum_{t'} m_{tt'} x_{2i}(t') \\ &\quad + \sum_{t'} m_{tt'} \sum_i u_i(t') \,. \end{aligned}$$

It is then easily verified that $\sum_{t'} m_{tt'} = 0$,[14] that $\sum_{t'} m_{tt'} x_{1i}(t')$ and $\sum_{t'} m_{tt'} x_{2i}(t')$ are equal to the residuals $V_{1i}(t)$ and $V_{2i}(t)$, respectively, of the auxiliary regressions (2.4), and that the last term $[\sum m_{tt'} \sum u_i(t')]$ is the t-th residual of the least-squares regression of the sum of the microdisturbances on the explanatory macrovariables x_1 and x_2. Hence we can write

$$\begin{aligned} v(t) &= \sum_i \beta_{1i} V_{1i}(t) + \sum_i \beta_{2i} V_{2i}(t) + \sum_{t'} m_{tt'} \sum_i u_i(t') \\ &= \sum_i u_i(t) + \sum_i \beta_{1i} V_{1i}(t) + \sum_i \beta_{2i} V_{2i}(t) + \sum_{t'} (m_{tt'} - \delta_{tt'}) \sum_i u_i(t') \,, \end{aligned} \tag{6.3}$$

[14] Proof: $\sum m_{tt'}$ is the first component of Mi (cf. footnote 10) and hence the leading element of MX, which is a zero matrix.

where $\delta_{tt'}$ is the Kronecker delta ($=1$ if $t=t'$, $=0$ if $t\neq t'$). We shall interpret the sum of the corresponding microdisturbances, $\sum u_i(t)$, as the "true" macrodisturbance (on the analogy of $\sum \alpha_i$ which is the "true value" of the macro-intercept), and

$$u(t)=\sum_i u_i(t)+\sum_i \beta_{1i}V_{1i}(t)+\sum_i \beta_{2i}V_{2i}(t) \tag{6.4}$$

as the macrodisturbance itself.[15] The aggregation bias is then the sum of the last two terms on the right of (6.4). Hence

$$v(t)=u(t)+\sum_{t'}(m_{tt'}-\delta_{tt'})\sum_i u_i(t')\,, \tag{6.5}$$

the last term on the right being the implied sampling error of the estimated macrodisturbance. It will be observed that this sampling error can be interpreted as the inner product of two vectors of T components, the first of which is the t-th row of

$$M-I=-X(X'X)^{-1}X'\,.$$

Disregarding the negative sign on the right, we note that this matrix produces the "systematic part" of regressions on X. For whatever column z of appropriate order, $X(X'X)^{-1}X'z$ is the systematic part of the regression of z on X. So we may conclude from (6.5) that, apart from sign, the implied sampling error of the estimate of the t-th macrodisturbance is equal to the systematic part in the t-th period of the least-squares regression of the sum of the microdisturbances on the explanatory macrovariables x_1 and x_2.

The results are summarized numerically in Table 6. We observe that the aggregation bias is usually smaller in absolute value than the "true value" of the macrodisturbance, but that it is far from being negligible; also, the two components of this bias have opposite signs (and hence compensate each other) in eleven cases out of twenty.

7. THE LEAST-SQUARES ESTIMATED MACRODISTURBANCES: (ii) VARIANCE AND FIRST-ORDER AUTOCOVARIANCE

Whereas we considered in the preceding section the components of the individual macrodisturbances, we shall now turn to parameters

[15] The result (6.4) has also been given in [5, pp. 110 seq.]; it is in accordance with the definition of the macrodisturbance as the discrepancy between the dependent macrovariable and the linear combination of the explanatory macrovariables whose weights are defined as indicated in the third paragraph of Section 2. It is further interesting to note that the sum of the $V_{1i}(t)$ and $V_{2i}(t)$ over all i is zero. For example, taking $1_i(\,)$ and summing over i, we obtain $\sum_{t'} m_{tt'}x_1(t')$, which is an element of the zero matrix of the preceding footnote.

TAELE 6

AGGREGATION BIAS AND IMPLIED SAMPLING ERRORS OF THE ESTIMATED MACRODISTURBANCES

Year	"True value" of macrodisturbance	Aggregation bias:			Macrodisturbance	Implied sampling error	Estimated macrodisturbance
		Total	due to β_{1i}	due to β_{2i}			
1935	107.0	62.9	5.3	57.6	169.9	25.5	195.4
36	24.7	11.1	−14.7	25.8	35.8	47.9	83.7
37	−130.2	17.7	−.6	18.3	−112.5	59.6	−52.8
38	−127.4	15.1	−46.5	61.5	−112.3	16.6	−95.8
39	−382.6	14.6	−22.9	37.1	−368.0	28.6	−339.4
40	−124.9	19.5	6.7	12.8	−105.4	31.5	−73.9
41	109.5	55.9	47.3	8.6	165.4	24.1	189.5
42	113.3	37.4	27.5	9.9	150.7	3.6	154.3
43	−26.3	−16.2	6.7	−22.9	−42.5	9.0	−33.5
44	4.9	−46.8	5.2	−51.0	−41.9	11.6	−30.2
45	−56.5	−89.3	−13.3	−75.9	−145.8	17.5	−128.3
46	214.1	−89.9	−8.5	−81.2	124.2	18.4	142.8
47	120.9	−20.8	−.9	−19.9	100.1	−18.1	82.0
48	140.3	−37.1	−24.8	−12.4	103.2	−29.9	73.3
49	−144.4	−4.8	8.3	−13.1	−149.2	−37.0	−186.2
50	−122.3	−19.5	−1.0	−18.4	−141.8	−37.7	−179.5
51	30.6	16.0	47.9	−31.8	46.6	−27.3	19.3
52	93.8	8.1	15.5	−7.4	101.9	−38.2	63.7
53	170.5	22.8	−5.1	27.9	193.3	−41.9	151.4
54	−15.1	43.1	−32.1	75.2	28.0	−63.7	−35.7

describing their statistical distribution. Their mean is not very interesting, because all terms entering equation (6.3) have zero average over time; hence the same thing must apply to the error terms themselves. The variance is more interesting, however. We shall analyze it by means of the following splitting-up:

$$v(t) = U_i + V_i + W_i \,, \tag{7.1}$$

where U_i is the "true value" of the macrodisturbance:

$$U_i = \sum_i u_i(t) \,, \tag{7.2}$$

V_i the aggregation bias:

$$V_i = \sum_i \beta_{1i} V_{1i}(t) + \sum_i \beta_{2i} V_{2i}(t) \,, \tag{7.3}$$

and W_i the implied sampling error:

$$W_i = \sum_{i'} (m_{ii'} - \delta_{ii'}) \sum_i u_i(t') \,. \tag{7.4}$$

The variance of the estimated macrodisturbance is then[18]

$$\frac{1}{T}\sum_t v(t)^2 = \frac{1}{T}\sum_t U_t^2 + \frac{1}{T}\sum_t V_t^2 + \frac{1}{T}\sum_t W_t^2 + \frac{2}{T}\sum_t U_t V_t + \frac{2}{T}\sum_t U_t W_t + \frac{2}{T}\sum_t V_t W_t , \tag{7.5}$$

i.e., this variance can be regarded as the sum of three variances and six covariances, the latter being pairwise equal. These components are shown numerically in Table 7; our comments are as follows:

(i) The first term, $(1/T)\sum U_t^2$, which is the variance of the true values of the macrodisturbances, is by far the largest. A consequence of this is that the true values of the macrodisturbances (U_t), the actual macrodisturbances including aggregation bias ($U_t + V_t$), and the least-squares estimated macrodisturbances ($U_t + V_t + W_t$) are highly correlated, the correlations ranging from 0.92 to 0.97. It is also interesting to note that the variance of the true values can be split up into two terms, one of which is the sum of the variances of the microdisturbances, the other being the sum of their covariances:

$$\frac{1}{T}\sum_t U_t^2 = \sum_i \frac{1}{T}\sum_t u_i(t)^2 + \sum_i \sum_{j \neq i} \frac{1}{T}\sum_t u_i(t)u_j(t) . \tag{7.6}$$

This splitting-up is presented numerically in Table 7, which shows that the sum of the microcovariances represents a small but positive contribution to the variance of the true values. This is the result of a tendency towards positive correlation of the microdisturbances corresponding to separate firms, a feature which suggests that the individual firms' investments were subject to a common factor that is not introduced explicitly as an explanatory variable in the microregressions. A survey of these correlations is given in Table 8; about two thirds of them are positive.

(ii) The second term, $(1/T)\sum V_t^2$, which is the variance of the aggregation bias, is almost 10 per cent of the first term. As the aggregation bias is the sum of two terms, one of which deals with β_{1i} and $V_{1i}(t)$ and the other with β_{2i} and $V_{2i}(t)$, we can make a further splitting-up according to

$$\frac{1}{T}\sum_t V_t^2 = \frac{1}{T}\sum_t \{\sum_i \beta_{1i} V_{1i}(t)\}^2 + \frac{1}{T}\sum_t \{\sum_i \beta_{2i} V_{2i}(t)\}^2 \tag{7.7}$$

[18] In order to avoid cumbersome expressions, we shall frequently write "variance" in what follows, instead of the term "sample variance" which is, strictly speaking, more correct.

TABLE 7

COMPOSITION OF THE VARIANCE OF THE ESTIMATED MACRODISTURBANCES

$(1/T)\sum U_t^2$	19547
sum of microvariances	16237
sum of microcovariances	3310
$(1/T)\sum V_t^2$	1655
variance of $\sum_t \beta_{1t} V_{1t}(t)$	528
variance of $\sum_t \beta_{2t} V_{2t}(t)$	1699
twice the covariance	−571
$(1/T)\sum W_t^2$	1103
$(2/T)\sum U_t V_t$	−900
$(2/T)\sum U_t W_t$	−2207
$(2/T)\sum V_t W_t$	0
Total	19198

$$+\frac{2}{T}\sum_t\{\sum_t \beta_{1t} V_{1t}(t)\}\{\sum_t \beta_{2t} V_{2t}(t)\}\ .$$

Table 7 shows that the second term is about three or four times as large as the first, and that the third term is negative. The latter feature implies that the two components of the aggregation bias tend to compensate each other, the former that the component referring to the capital stock of the firm dominates the other one.

(iii) The third term, $(1/T)\sum W_t^2$, which is the variance of the implied sampling errors, is about 20–30 per cent below the value of the second. It is of some interest to consider this term in more detail:

$$\begin{aligned}\sum W_t^2 &= \sum_t\{\sum_{t'}(m_{tt'}-\delta_{tt'})\sum_t u_t(t')\}^2 \\ &= \sum_t\{\sum_{t'}(m_{tt'}-\delta_{tt'})\sum_{t''}(m_{tt''}-\delta_{tt''})\sum_t\sum_\tau u_t(t')u_\tau(t'')\}\ .\end{aligned} \tag{7.8}$$

Assuming that there are no lagged correlations among the microdisturbances, and that the current covariances are independent of time,

$$E[u_t(t')u_\tau(t'')] = \begin{cases}\sigma_{t\tau} & \text{if } t'=t'' \\ 0 & \text{if } t'\neq t''\ ,\end{cases} \tag{7.9}$$

we obtain for the expectation of $\sum W_t^2$

$$\begin{aligned}E(\sum W_t^2) &= (\sum_t\sum_\tau \sigma_{t\tau})\sum_t\sum_{t'}(m_{tt'}-\delta_{tt'})^2 \\ &= (\sum\sum \sigma_{t\tau})\,\mathrm{tr}\,(M-I)(M-I)' \\ &= (\sum\sum \sigma_{t\tau})\,\mathrm{tr}\,X(X'X)^{-1}X'\end{aligned} \tag{7.10}$$

TABLE 8

CORRELATION MATRIX OF THE MICRODISTURBANCES

	G.M.	U.S. Steel	G. Elec.	Chrys.	Atl. Ref.	I.B.M.	U. Oil	West	G. Year	D. Match
G.M.	1	−.26	.28	−.27	−.31	.12	.51	.15	.21	−.26
U.S. Steel		1	.43	.34	.22	.36	−.27	.61	.28	.70
G. Elec.			1	−.07	−.02	.46	−.02	.73	.41	.56
Chrys.				1	.06	.20	−.12	.12	.07	.11
Atl. Ref.					1	.21	.15	.00	−.14	.12
I.B.M.						1	.13	.52	−.17	.39
U. Oil							1	.14	.19	−.22
West.								1	.52	.57
G. Year									1	.26
D. Match										1

$$E(\sum W_i^2) = (\sum\sum \sigma_{ij}) \operatorname{tr} (X'X)^{-1}X'X = A\sum\sum \sigma_{ij},$$

where use has been made of the fact that the sum of squares of all elements of some matrix A is the trace of AA';[17] further, A is the number of columns of X (3 in our case). So we can conclude that, under assumption (7.9), the expectation of the third term is $(3/20)\sum\sum \sigma_{ij}$. The double sum $\sum\sum \sigma_{ij}$ is unknown, of course, but it can be estimated without bias by the estimated variance $(1/T)\sum U_i^2$ of the true values of the macrodisturbances in view of the definition (7.2). Comparing this with the figures of Table 7, we find that the estimated expectation of the third term is about 3000 and that the observed value is less than one half of this.

(iv) The fourth term, $(2/T)\sum U_t V_t$, which is twice the covariance of the true values and the aggregation bias, is negative under both assumptions. As is easily verified, the expected value of this term is zero when V_t is nonstochastic and U_t has zero expectation. It is of some interest to pursue this a little further, because if the observed value of this term were significantly different from zero, this would point to a dependence between explanatory microvariables and microdisturbances, the reason being that the residuals V_t of the auxiliary regressions are nothing else than combinations of values taken by the explanatory microvariables, while U_t is a combination of microdisturbances. There is, of course, nothing which prevents a dependence of this kind, even if the disturbances of microequations are all independent of the explanatory variables of the same equation, because V_t

[17] The trace of a matrix is the sum of the diagonal elements. In deriving (7.10), use has also been made of the fact that tr AB = tr BA (provided that both product matrices

and U_t are based on all explanatory microvariables and all microdisturbances at time t. Now the significance of the present term can be easily judged by computing the variance of $\sum U_t V_t$:

$$\begin{aligned} \operatorname{var}(\sum U_t V_t) &= \sum_t \sum_{t'} V_t V_{t'} \sum_i \sum_j E[u_i(t) u_j(t')] \\ &= (\sum \sum \sigma_{ij}) \sum_t V_t^2 , \end{aligned} \tag{7.11}$$

where use is made of assumption (7.9) (and of course also of the independence of the U's and the V's, which is our null-hypothesis here). It follows that $(2/T) \sum U_t V_t$ is distributed with zero mean and standard deviation

$$\frac{2}{T} \{(\sum \sum \sigma_{ij}) \sum_t V_t^2\}^{\frac{1}{2}} ,$$

which is estimated as about 2600. The observed values are therefore not significantly different from zero.

(v) The last two terms are very simple. The sixth, $(2/T) \sum V_t W_t$, is necessarily zero, which means that aggregation bias and implied sampling errors of least-squares estimated macrodisturbances are always uncorrelated in the sample. This is proved as follows. We write

$$\sum V_t W_t = \sum_t \sum_{t'} u_i(t') \sum_t (m_{tt'} - \delta_{tt'}) V_t ,$$

the last part of which is

$$\sum_t (m_{tt'} - \delta_{tt'}) \sum_j V_{1j}(t) \beta_{1j} + \sum_t (m_{tt'} - \delta_{tt'}) \sum_j V_{2j}(t) \beta_{2j} .$$

Both terms vanish, because

$$\sum_t (m_{tt'} - \delta_{tt'}) V_{1j}(t) = \sum_t (m_{tt'} - \delta_{tt'}) V_{2j}(t) = 0 .$$

This follows from the fact that these expressions are equal to minus the systematic parts at time t of the regressions of V_{1j} and V_{2j} on the explanatory macrovariables, which must be zero because V_{1j} and V_{2j} are least-squares residuals corresponding to these explanatory variables (cf. the end of the third papagraph).

The fifth term, $(2/T) \sum U_t W_t$, is necessarily twice the third except for sign. This means that the covariance of the true values and the implied sampling errors is always minus the variance of these errors, which is proved as follows. For T times the covariance we can write

$$\begin{aligned} \sum U_t W_t &= \sum_t \sum_t u_i(t) \sum_{t'} (m_{tt'} - \delta_{tt'}) \sum_j u_j(t') \\ &= \sum_t \sum_{t'} (m_{tt'} - \delta_{tt'}) U_t U_{t'} , \end{aligned} \tag{7.12}$$

and for T times the variance of the implied sampling errors:

$$\begin{aligned}\sum_i W_i^2 &= \sum_t \sum_{t'} \sum_{t''} (m_{tt'} - \delta_{tt'})(m_{tt''} - \delta_{tt''}) \sum_i \sum_j u_i(t')u_j(t'') \\ &= \sum_{t'} \sum_{t''} U_{t'}U_{t''} \sum_t (m_{tt'} - \delta_{tt'})(m_{tt''} - \delta_{tt''}) .\end{aligned} \tag{7.13}$$

But for the last sum we have

$$\sum_t (m_{tt'} - \delta_{tt'})(m_{tt''} - \delta_{tt''}) = -(m_{t't''} - \delta_{t't''}) , \tag{7.14}$$

because the left-hand side is equal to the (t', t'') element of the matrix $(M - I)^2 = -(M - I)$. Combining (7.12), (7.13) and (7.14), we find that our assertion has been proved.

Finally, we turn to the first-order autocovariance, the importance of which will be clear in view of the fact that this covariance should vanish according to the classical conditions of regression analysis. We have

$$\begin{aligned}\frac{1}{T-1} \sum_t v(t)v(t-1) = \frac{1}{T-1} [&\sum U_t U_{t-1} + \sum U_t V_{t-1} \\ &+ \sum U_t W_{t-1} + \sum V_t U_{t-1} + \sum V_t V_{t-1} + \sum V_t W_{t-1} \\ &+ \sum W_t U_{t-1} + \sum W_t V_{t-1} + \sum W_t W_{t-1}] .\end{aligned} \tag{7.15}$$

The autocovariances, as well as the separate terms on the right of (7.15), are shown in Table 9. For the least-squares estimated macro-disturbances the autocovariance is about 0.3 times the variance, implying a first-order autocorrelation equal to that fraction. It is further seen that the first component, $\{1/(T-1)\} \sum U_t U_{t-1}$, is by far the largest under both assumptions; this is in spite of the fact that it has zero expectation under the conditions just-mentioned. To test these conditions, we consider the variance of $\sum U_t U_{t-1}$:

$$\begin{aligned}E(\sum_t U_t U_{t-1})^2 &= \sum_t \sum_{t'} \sum_i \sum_j \sum_k \sum_l E[u_i(t)u_j(t-1)u_k(t')u_l(t'-1)] \\ &= \sum_t \sum_{t'} \sum_j \sum_k \sum_l E[u_i(t)u_k(t)]E[u_j(t-1)u_l(t-1)] \\ &= (T-1)(\sum \sum \sigma_{ij})^2 ,\end{aligned} \tag{7.16}$$

which implies that—under the null-hypothesis—, $\{1/(T-1)\} \sum U_t U_{t-1}$ is distributed with zero mean and standard deviation

$$\frac{1}{\sqrt{T-1}} \sum \sum \sigma_{ij} ,$$

this standard deviation being estimated as slightly more than 4500. Comparing this with the observed values of Table 9, we can conclude

TABLE 9

COMPOSITION OF THE FIRST-ORDER AUTOCOVARIANCE OF THE ESTIMATED MACRODISTURBANCES

	U_t	V_t	W_t
U_{t-1}	7820	−396	−1787
V_{t-1}	−1299	1109	271
W_{t-1}	−789	3	895
Total		5827	

that they are not significantly positive.

The other eight components of the autocovariance are much smaller, and four of them have zero expectation. The other four are $\sum V_t V_{t-1}$ (which is nonstochastic), $\sum U_t W_{t-1}$, $\sum W_t U_{t-1}$ and $\sum W_t W_{t-1}$. The expectations of the last three terms are derived as follows:

$$E(\sum_t U_t W_{t-1}) = E\sum_t \{\sum_i u_i(t) \sum_{t'} (m_{t-1,t'} - \delta_{t-1,t'}) \sum_j u_j(t')\}$$
$$= (\sum\sum \sigma_{ij}) \sum_t (m_{t-1,t} - \delta_{t-1,t}) = (\sum\sum \sigma_{ij}) \sum_t m_{t-1,t}\,; \tag{7.17}$$

$$E(\sum_t W_t U_{t-1}) = E\sum_t \{\sum_i u_i(t-1) \sum_{t'} (m_{tt'} - \delta_{tt'}) \sum_j u_j(t')\}$$
$$= (\sum\sum \sigma_{ij}) \sum_t m_{t,t-1}\,; \tag{7.18}$$

$$E(\sum_t W_t W_{t-1})$$
$$= E\sum_t \{\sum_{t'} (m_{tt'} - \delta_{tt'}) \sum_{t''} (m_{t-1,t''} - \delta_{t-1,t''}) \sum\sum_j u_i(t') u_j(t'')\}$$
$$= (\sum\sum \sigma_{ij}) \sum_t \sum_{t'} (m_{tt'} - \delta_{tt'})(m_{t-1,t'} - \delta_{t-1,t'})$$
$$= -(\sum\sum \sigma_{ij}) \sum_t m_{t,t-1}\,. \tag{7.19}$$

Hence all three expectations are equal except that the last has opposite sign. Taking account of the factor $1/(T-1)$ that was applied in Table 9, we find that the expectations corresponding to (7.17) and (7.18) are about −2000, and the one corresponding to (7.19) about 2000. With only one exception, the observed values of Table 9 correspond with these expectations as far as signs are concerned.

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APPENDIX

The data underlying this study, taken from Grunfeld [3], are presented below in Table 10. The exact definitions of these variables are as follows.

I = Gross investment[1] = additions to plant and equipment plus maintenance and repairs in millions of dollars deflated by P_1.

F = Value of the firm[2] = price of common and preferred shares at December 31 (or average price of December 31 and January 31 of the following year) times number of common and preferred shares outstanding plus total book value of debt at December 31 in millions of dollars deflated by P_2.

C = The stock of plant and equipment[1] = accumulated sum of net additions to plant and equipment deflated by P_1 minus depreciation allowance deflated by P_3.

In these definitions

P_1 = Implicit price deflator of producers durable equipment (base 1947).[3]

P_2 = Implicit price deflator of G.N.P. (base 1947).[3]

P_3 = Depreciation expense deflator = ten years moving average of wholesale price index of metals and metal products (base 1947).[4]

A splitting-up of the implied sampling errors of a, b_1 and b_2 by firms and years is given in Tables 11-13.

TABLE 10

MICRODATA UNDERLYING THIS STUDY

	G.M.			U.S. Steel			G. Elec.			Chrys.		
	I	F_{-1}	C_{-1}	I	F_{-1}	C_{-1}	I	F_{-1}	C_{-1}	I	F_{-1}	C_{-1}
1935	317.6	3078.5	2.8	209.9	1362.4	53.8	33.1	1170.6	97.8	40.29	417.5	10.5
36	391.8	4661.7	52.6	355.3	1807.1	50.5	45.0	2015.8	104.4	72.76	837.8	10.2
37	410.6	5387.1	156.9	469.9	2676.3	118.1	77.2	2803.3	118.0	66.26	883.9	34.7
38	257.7	2792.2	209.2	262.3	1801.9	260.2	44.6	2039.7	156.2	51.60	437.9	51.8
39	330.8	4313.2	203.4	230.4	1957.3	312.7	48.1	2256.2	172.6	52.41	679.7	64.3
40	461.2	4643.9	207.2	361.6	2202.9	254.2	74.4	2132.2	186.6	69.41	727.8	67.1
41	512.0	4551.2	255.2	472.8	2380.5	261.4	113.0	1834.1	220.9	68.35	643.6	75.2
42	448.0	3244.1	303.7	445.6	2168.6	298.7	91.9	1588.0	287.8	46.80	410.9	71.4
43	499.6	4053.7	264.1	361.6	1985.1	301.8	61.3	1749.4	319.9	47.40	588.4	67.1
44	547.5	4379.3	201.6	288.2	1813.9	279.1	56.8	1687.2	321.3	59.57	698.4	60.5
45	561.2	4840.9	265.0	258.7	1850.2	213.8	93.6	2007.7	319.6	88.78	846.4	54.6
46	688.1	4900.9	402.2	420.3	2067.7	132.6	159.9	2208.3	346.0	74.12	893.8	84.8
47	568.9	3526.5	761.5	420.5	1796.7	264.8	147.2	1656.7	456.4	62.68	579.0	96.8
48	529.2	3254.7	922.4	494.5	1625.8	306.9	146.3	1604.4	543.4	89.36	694.6	110.2
49	555.1	3700.2	1020.1	405.1	1667.0	351.1	98.3	1431.8	618.3	78.98	590.3	147.4
50	642.9	3755.6	1099.0	418.8	1677.4	357.8	93.5	1610.5	647.4	100.66	693.5	163.2
51	755.9	4833.0	1207.7	588.2	2289.5	342.1	135.2	1819.4	671.3	160.62	809.0	203.5
52	891.2	4924.9	1430.5	645.5	2159.4	444.2	157.3	2079.7	726.1	145.00	727.0	290.6
53	1304.4	6241.7	1777.3	641.0	2031.3	623.6	179.5	2371.6	800.3	174.93	1001.5	346.1
54	1486.7	5593.6	2226.3	459.3	2115.5	669.7	189.6	2759.9	888.9	172.49	703.2	414.9

[1] Source: *Moody's Industrial Manual* and Annual Reports of Corporations.

[2] Source: *Bank and Quotation Record* and *Moody's Industrial Manual*.

[3] Source: *Survey of Current Business*, July 1956 and July 1957.

[4] Source: *Historical Statistics of the U.S.* 1789-1945 and *Economic Report of the President*, January 1957, p. 161.

TABLE 10 (*concluded*)

	Atl. Ref.			I.B.M.			U. Oil		
	I	F_{-1}	C_{-1}	I	F_{-1}	C_{-1}	I	F_{-1}	C_{-1}
1935	39.68	157.7	183.2	20.36	197.0	6.5	24.43	138.0	100.2
36	50.73	167.9	204.0	25.98	210.3	15.8	23.21	200.1	125.0
37	74.24	192.9	236.0	25.94	223.1	27.7	32.78	210.1	142.4
38	53.51	156.7	291.7	27.53	216.7	39.2	32.54	161.2	165.1
39	42.65	191.4	323.1	24.60	286.4	48.6	26.65	161.7	194.8
40	46.48	185.5	344.0	28.54	298.0	52.5	33.71	145.1	222.9
41	61.40	199.6	367.7	43.41	276.9	61.5	43.50	110.6	252.1
42	39.67	189.5	407.2	42.81	272.6	80.5	34.46	98.1	276.3
43	62.24	151.2	426.6	27.84	287.4	94.4	44.28	108.8	300.3
44	52.32	187.7	470.0	32.60	330.3	92.6	70.80	118.2	318.2
45	63.21	214.7	499.2	39.03	324.4	92.3	44.12	126.5	336.2
46	59.37	232.9	534.6	50.17	401.9	94.2	48.98	156.7	351.2
47	58.02	249.0	566.6	51.85	407.4	111.4	48.51	119.4	373.6
48	70.34	224.5	595.3	64.03	409.2	127.4	50.00	129.1	389.4
49	67.42	237.3	631.4	68.16	482.2	149.3	50.59	134.8	406.7
50	55.74	240.1	662.3	77.34	673.8	164.4	42.53	140.8	429.5
51	80.30	327.3	683.9	95.30	676.9	177.2	64.77	179.0	450.6
52	85.40	359.4	729.3	99.49	702.0	200.0	72.68	178.1	466.9
53	81.90	398.4	774.3	127.52	793.5	211.5	73.86	186.8	486.2
54	81.43	365.7	804.9	135.72	927.3	238.7	89.51	192.7	511.3

	West.			G. Year			D. Match		
	I	F_{-1}	C_{-1}	I	F_{-1}	C_{-1}	I	F_{-1}	C_{-1}
1935	12.93	191.5	1.8	26.63	290.6	162	2.54	70.91	4.50
36	25.90	516.0	.8	23.39	291.1	174	2.00	87.94	4.71
37	35.05	729.0	7.4	30.65	335.0	183	2.19	82.20	4.57
38	22.89	560.4	18.1	20.89	246.0	198	1.99	58.72	4.56
39	18.84	519.9	23.5	28.78	356.2	208	2.03	80.54	4.38
40	28.57	628.5	26.5	26.93	289.8	223	1.81	86.47	4.21
41	48.51	537.1	36.2	32.08	268.2	234	2.14	77.68	4.12
42	43.34	561.2	60.8	32.21	213.3	248	1.86	62.16	3.83
43	37.02	617.2	84.4	35.69	348.2	274	.93	62.24	3.58
44	37.81	626.7	91.2	62.47	374.2	282	1.18	61.82	3.41
45	39.27	737.2	92.4	52.32	387.2	316	1.36	65.85	3.31
46	53.46	760.5	86.0	56.95	347.4	302	2.24	69.54	3.23
47	55.56	581.4	111.1	54.32	291.9	333	3.81	64.97	3.90
48	49.56	662.3	130.6	40.53	297.2	359	5.66	68.00	5.38
49	32.04	583.8	141.8	32.54	276.9	370	4.21	71.24	7.39
50	32.24	635.2	136.7	43.48	274.6	376	3.42	69.05	8.74
51	54.38	723.8	129.7	56.49	339.9	391	4.67	83.04	9.07
52	71.78	864.1	145.5	65.98	474.8	414	6.00	74.42	9.93
53	90.08	1193.5	174.8	66.11	496.0	443	6.53	63.51	11.68
54	68.60	1188.9	213.5	49.34	474.5	468	5.12	58.12	14.33

TABLE 11

COMPONENTS OF THE IMPLIED SAMPLING ERROR OF THE CONSTANT TERM BY FIRMS AND YEARS*

	G.M.	U.S. Steel	G. Elec.	Chrys.	Atl. Ref.	I.B.M.	U. Oil	West.	G. Year	D. Match	Total
1935	5471	−5	−158	586	−508	143	245	173	−47	5	5905
36	155	−315	66	−48	−1	−26	24	4	23	3	−113
37	6657	−242	245	351	−923	−139	−60	174	90	16	6169
38	−119	−3812	−846	265	162	159	91	−287	−225	−16	−4629
39	75	127	19	10	8	6	5	7	5	0	263
40	184	679	5	21	69	59	19	61	51	5	1154
41	−21	−3	−36	−1	−5	−9	−6	−15	−0	0	−97
42	3335	−27	546	−54	−513	299	−130	290	119	−9	3856
43	1082	−861	−379	−215	218	−149	33	−47	−85	−17	−419
44	1060	−938	−284	−82	−25	−106	271	−35	200	−8	52
45	−255	718	−13	−85	−30	20	30	56	−36	4	410
46	−1757	−952	−993	273	48	34	63	−98	−231	−6	−3620
47	480	1655	1388	−216	−217	−80	−116	474	401	52	3821
48	−1999	5389	1203	255	360	310	−193	113	−140	109	5408
49	−4262	961	−870	−273	156	26	−260	−424	−406	18	−5334
50	−1796	998	−1061	36	−226	−469	−522	−383	−10	−25	−3458
51	967	−841	40	−320	−19	1	18	−41	−53	−1	−248
52	989	−1830	−24	−35	−26	15	−48	−166	−50	−15	−1191
53	−2230	−4145	−229	280	−590	−623	−82	−508	−3	−44	−7679
54	−4931	4239	297	247	109	−74	−480	469	594	55	525
Total	3085	794	−1083	996	−1457	−602	−1100	−182	197	127	775

* All figures are to be divided by 100.

TABLE 12

COMPONENTS OF THE IMPLIED SAMPLING ERROR OF THE F_{-1}-COEFFICIENT BY FIRMS AND YEARS*

	G.M.	U.S. Steel	G. Elec.	Chrys.	Atl. Ref.	I.B.M.	U. Oil	West.	G. Year	D. Match	Total
1935	−4230	4	122	−453	393	−111	−190	−133	37	−4	−4565
36	−767	1553	−325	235	3	128	−119	−21	−115	−14	557
37	−9566	348	−352	−504	1326	200	86	−250	−130	−24	−8865
38	87	2775	616	−193	−118	−116	−66	209	164	11	3369
39	−1471	−2479	−377	−197	−163	−114	−96	−139	−100	−6	−5141
40	−452	−1666	−13	−53	−168	−146	−47	−150	−125	−13	−2832
41	299	48	505	8	64	129	88	214	5	−2	1358
42	−2718	22	−445	44	418	−244	106	−236	−97	7	−3143
43	−568	452	199	113	−115	78	−17	25	45	9	220
44	−237	210	63	18	6	24	−61	8	−45	2	−12
45	594	−1672	30	199	69	−47	−69	−131	83	−9	−955
46	2773	1502	1568	−431	−75	−54	−99	154	365	9	5713
47	−474	−1636	−1373	213	215	79	115	−469	−397	−51	−3778
48	2144	−5781	−1291	−273	−386	−333	207	−121	150	−117	−5801
49	4809	−1084	982	308	−176	−30	294	479	458	−20	6019
50	2116	−1176	1250	−42	267	553	615	451	12	29	4075
51	−657	571	−27	217	13	−1	−12	28	36	1	168
52	−550	1018	13	20	15	−8	27	93	28	8	663
53	1883	3501	194	−236	80	526	69	429	3	37	6485
54	3103	−2667	−187	−155	−69	46	302	−295	−373	−34	−330
Total	−3881	−6158	1152	−1164	1598	560	1134	142	2	−181	−6795

* All figures are to be divided by 10^6.

TABLE 13

COMPONENTS OF THE IMPLIED SAMPLING ERROR OF THE C_{-1}-COEFFICIENT BY FIRMS AND YEARS*

	G.M.	U.S. Steel	G. Elec.	Chrys.	Atl. Ref.	I.B.M.	U. Oil	West.	G. Year	D. Match	Total
1935	−1449	1	42	−155	135	−38	−65	−46	13	−1	−1565
36	1827	−3699	774	−560	−7	−305	284	51	275	34	−1326
37	10825	−394	339	570	−1501	−226	−97	283	147	27	10032
38	32	1037	230	−72	−44	−43	−25	78	61	4	1259
39	3509	5914	899	471	389	273	228	331	238	13	12265
40	742	2738	21	86	277	240	78	247	205	22	4656
41	−657	−106	−1110	−17	−140	−284	−194	−470	−10	5	−2949
42	345	−3	56	−6	−53	31	−13	30	12	−1	399
43	−464	369	163	92	−94	64	−14	20	37	7	180
44	−1100	973	294	86	26	110	−281	36	−207	8	−54
45	−767	2159	−39	−257	−89	61	89	170	−107	12	1233
46	−2620	−1419	−1481	407	71	51	94	−146	−345	−9	−5399
47	396	1365	1146	−178	−179	−66	−96	391	331	43	3153
48	−2101	5665	1265	268	378	326	−203	119	−147	115	5685
49	−5495	1239	−1122	−351	201	34	−336	−547	−523	23	−6878
50	−2937	1632	−1735	59	−370	−767	−854	−626	−17	−41	−5656
51	−3093	2688	−129	1023	60	−3	−56	130	168	4	792
52	−2836	5250	69	101	76	−43	138	478	143	42	3417
53	1595	2966	164	−200	68	446	59	364	2	31	5495
54	8283	−7121	−498	−415	−184	124	806	−789	−997	−92	−881
Total	4035	21255	−593	986	−982	−16	−459	104	−722	248	23856

* All figures are to be divided by 10^6.

REFERENCES

[1] BALDERSTON, J. B., AND T. M. WHITIN: "Aggregation in the Input-Output Model," *Economic Activity Analysis*, ed. by O. Morgenstern (New York: John Wiley and Sons, 1954) pp. 79–128.

[2] GRILICHES, Z.: "Distributed Lags, Disaggregation and Regional Demand Functions for Fertilizers," University of Chicago, Office of Agriculture Economics Research, Paper nr. 5820 (Chicago, Ill.: September, 1958).

[3] GRUNFELD, Y.: *The Determinants of Corporate Investment*. Unpublished Ph. D. Thesis (Chicago, 1958).

[4] GRUNFELD, Y., AND Z. GRILICHES: "Is Aggregation Necessarily Bad?" Mimeographed and unpublished (Chicago, September 1958).

[5] THEIL, H.: *Linear Aggregation of Economic Relations*. (Amsterdam: North Holland Publishing Company, 1954).

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THE ST. PETERSBURG PARADOX AS A DIVERGENT DOUBLE LIMIT*

BY PAUL A. SAMUELSON

1. OFTEN, to analyze a paradox is to dispel it. The St. Petersburg Paradox, in which Paul refuses to pay Peter the extraordinarily high price that ordinary calculation of mathematical expectation of money-gain computes for the privilege of Paul's receiving 2^i ducats if heads first appears at the i-th toss of a fair coin, made an important contribution to the history of philosophy and probability. It dramatized the fact that men feel they value money losses and gains at something different from their expected or arithmetic-mean values. Had all Pauls not already instinctively felt this, they would not have recoiled in shock from the prospect of paying Peter the infinite true mathematical odds, but it took the formulation of the problem by Cramer, Bernoulli, and others to rationalize what men already felt.[1]

Now it may be cogently argued that a long series of—say 1 billion—coin tosses is really sufficient to drive home the point that rational men maximize their "moral expectation" (i.e., expected value of utility of wealth) rather than maximizing their simple "money expectation". And yet, how much more dramatic is infinity than any large finite number!

A present-day savant might grant the eye-opening service rendered by the St. Petersburg Paradox and still argue that it has by now done its work, and that—like most of the scaffoldings used to construct the edifice called the modern mind—it can now be dispensed with. Accepting the fact that each man acts as if he has an ultimately bounded utility function, or finite wealth and limited borrowing power, or only a finite life span in which to toss coins, such sophisticates argue that the St. Petersburg Paradox can never arise in real life. *Paul will never be willing to give as much as Peter will demand for such a contract; and hence the indicated activity will take place at the equilibrium level of zero intensity.* Zero cancels out infinity, so to speak.

2. I do not disagree with this line of argument. But I propose to

* Manuscript received July, 14, 1959.

[1] I. Todhunter, *History of the Mathematical Theory of Probability* (New York: Chelsea, 1949) p. 220 gives a good account of the early discussions.

show here that the essence of the paradox—its infinity—remains even after we take into account the fact that marginal utility of money is not a constant. For, as (i) we make the constant size of the stake at each coin toss, y, get smaller and smaller, and (ii) let the stipulated number, n, of coin tosses get larger and larger, we find that the value, v, which *any* rational Paul will in real life pay for the privilege of playing becomes indefinitely large in ratio to the size of the stakes. That is, v/y must go to infinity as y goes to zero.

3. To avoid the non-operational concept of an infinity of tosses, I consider a finite game of n tosses: if heads never comes up, as will happen with probability 2^{-n}, Peter returns the value v Paul has paid him and all bets are off; if heads first turns up at the i-th toss, Peter pays Paul 2^i times the agreed stakes-parameter y. (I.e., y = one ducat, one dollar, one dime, or any stipulated positive amount.) To calculate v, the maximum that Paul will pay for the privilege of playing the game, we find what expected value of his utility $\sum_1^n 2^{-i}U(x-v+y2^i) + 2^{-n}U(x)$ will equal the certain utility $U(x)$ of not playing at all, thereby leaving him indifferent whether or not to play.

4. This is summarized in the following mathematical theorem about a double limit.

THEOREM: *Let utility be a smooth function of money so that* $U(X)$, $U'(X)$, $U''(X)$, *are continuous functions for* $X \geqq 0$; *Let* $U'(X) \geqq 0$; *and at* $X = x$ *where* $U'(x) > 0$, *let* $v = V(n, y; x)$ *be defined by the following implicit equation*

$$U(x) = \frac{1}{2}U(x - v + y2) + \frac{1}{4}U(x - v + y4) + \cdots + \frac{1}{2^n}U(x - v + y2^n) + \frac{1}{2^n}U(x) .$$

Then

$$\lim_{n\to\infty}\lim_{y\to 0}\frac{V(n, y; x)}{y} = \lim_{y\to 0}\lim_{n\to\infty}\frac{V(n, y; x)}{y} = \infty .$$

The intuitive proof of this is simple for the economist. As the size of the stake goes to zero, the marginal utility of money becomes sensibly constant so that the traditional formulation of ordinary expected value of money becomes applicable,[3] with

[3] Professor Robert M. Solow points out that for any prizes $(yZ_1, yZ_2, \cdots, yZ_n)$, occurring with respective probabilities $(p_1, p_2, \cdots, p_n)$, v/y will in the limit always equal the expected value of Z, $E(Z) = \sum p_i Z_i$. Later I refer to this as Solow's Theorem.

$$v = \frac{2}{2}y + \frac{4}{4}y + \frac{8}{8}y + \cdots + \frac{2^n}{2^n}y + \cdots = \infty .$$

In real life, mathematicians should perhaps have recognized that at most one can lay down one's all for anything, so that $v \leqq x$. But the infinity paradox remains in that as $x \to \infty$, the maximizer of mere money gain will pay his all regardless of how large that all becomes!

For a mathematical proof of the Theorem, we note that since $V(n, 0; x) = 0$,

$$\lim_{n\to\infty}\lim_{y\to 0}\frac{V(n, y; x)}{y} = \lim_{n\to\infty}\lim_{y\to 0}\frac{\partial V(n, y; x)}{\partial y}$$

$$(1) \qquad = \lim_{n\to\infty}\left.\frac{\sum_1^n 2^{-i}U'(x - v + y2^i)2^i}{\sum_1^n 2^{-i}U'(x - v + y2^i)}\right|_{y=0} \quad \text{by differentiation of our implicit equation,}$$

$$= \lim_{n\to\infty}\frac{\sum_1^n 1}{\sum_1^n 2^{-i}} = \lim_{n\to\infty}\frac{n}{1 - 2^{-n}} = \infty .$$

To prove the divergence of the double limit taken in the reverse order, note that with $V(n, 0; x) \equiv 0$ for all n and x, $V(\infty, 0; x) = 0$. Now consider $V(\infty, y; x)$ for any small y. If it is infinite, *a fortiori* our double limit is divergent. If it is finite, the limiting V/y must equal the limiting value of $\partial V/\partial y$ as $y \to 0$. Where $V(\infty, y; x)$ is finite,

$$(2) \qquad \frac{\partial V(\infty, y; x)}{\partial y} = \frac{\sum_1^\infty U'(x - v + y2^i)}{\sum_1^\infty 2^{-i}U'(x - v + y2^i)} .$$

But if v is finite, the denominator must be finite, since for $X \geqq$ some A, $U'(X) \leqq$ some B, and the denominator is dominated by a convergent geometric series. The numerator, however, must as $y \to 0$ approach $U'(x)\lim_{n\to\infty} n = \infty$[3].

[3] Professor Solow has contributed the following somewhat stronger result: The *general* double limit holds, entirely independently of the path along which the limits are taken,

$$\lim_{\substack{n\to\infty \\ y\to 0}}\frac{V}{y} = \infty, \text{ or equivalently } \lim_{\substack{n\to\infty \\ y\to 0}}\frac{y}{V(n, y; x)} = 0 .$$

This follows from results already deduced, supplemented by some slightly more delicate considerations. Consider first

$$\lim_{n\to\infty}\frac{y}{V(n, y; x)}$$

as a function of y (x is fixed). If we give this function the value 0 at $y = 0$, the arguments already made show it to be a continuous function of y in a closed interval

A geometrical description of our result may be of interest. Suppose $V(\infty, y; x)$ is plotted against the size of the stake y for any fixed wealth x. Then the original St. Petersburg paradox consisted of the recognition that V would be infinite for all positive y for an "expected-money" maximizer. But Bernoulli and others "resolved" the paradox by showing that an "expected-utility" maximizer, whose utility function was sufficiently well behaved, would have a finite $V(\infty, y; x)$ function for all y and, incidentally, one that begins at the origin.

What the present paper shows is essentially the following: even in the case where $V(\infty, y; x)$ is finite and enters the origin, it must enter the origin at a limiting 90° vertical angle—a strong result but not one that is, on reflection, paradoxical.

GENERALIZATION BEYOND CASE OF MORAL EXPECTATION MAXIMIZER

5. In concluding, I should like to indicate that the validity of our limit does not depend upon the willingness of a rational man to act so as to maximize moral expectation, i.e., to maximize the first moment of the utility of the possible outcomes $\sum p_i U(X_i)$ where $(X_1, X_2, \cdots; p_1, p_2, \cdots)$ represent the possible outcomes known to occur, with respective probabilities $(p_1, p_2, \cdots)$. There are persuasive reasons why a "rational man" might out of a sense of consistency alone wish to maximize such a moral expectation, as Marschak, Savage and Ramsey have shown[4]. But even if he does not so act, provided only that he maximizes an ordinal utility indicator $W(X_1, X_2, \cdots; p_1, p_2)$ that is smooth enough to have continuous second partial derivatives, our infinite limit for v/y will hold. The heuristic reasoning to demonstrate this follows.

6. Suppose Paul maximizes an ordinal utility function $W(X_1, X_2, \cdots; p_1, p_2, \cdots)$, such as $(\sum P_i X_i)^{1/2}(\sum P_i X_i^{-1})^{-3/2}$, which cannot be put into the additive form $\sum P_i U(X_i)$. Purely as a notational matter $(X_1, X_2, X_3, \cdots; p_1, p_2, p_3, \cdots)$ is the same thing as $(X_3, X_2, \cdots; p_1 + p_2, p_3, \cdots)$ if $X_1 = X_2$. So with W twice differentiable,

$0 \leqq y \leqq y^*$, and the limit of a sequence of continuous functions. Moreover, consideration of the game itself shows that $V(n, y; x)$ is a monotone increasing with n, so that y/V is decreasing. By Dini's Theorem on the convergence of monotone sequences of functions, $\lim_{n\to\infty} y/V$ exists uniformly. In turn, this uniform convergence together with what is proved about the repeated limits guarantees that the double limit

$$\lim_{\substack{n\to\infty \\ y\to 0}} \frac{y}{V} = 0 .$$

[4] Jacob Marschak, "Rational Behavior, Uncertain Prospects, and Measurable Utility," *Econometrica* Vol. 18, 1950, p. 111; L. J. Savage, *Foundations of Statistics* (New York: Wiley, 1954); F. P. Ramsey, *Foundations of Mathematics* (London: Routledge and Kegan Paul, 1930), Ch. VII.

$$(3)\quad \lim_{X_1 \to X_2} W(X_1, X_2, X_3, \cdots; p_1, p_2, p_3, \cdots) = W(X_2, X_3, \cdots; p_1+p_2, p_3, \cdots)\,,$$

from which it will be shown that for all X's close enough together, we can approximate W by the linear expression

$$(4)\quad W = W(\bar{X}; 1) + \frac{\partial W(\bar{X}; 1)}{\partial X} \sum_1^n p_i(X_i - \bar{X})\,.$$

7. This last expression asserts the fundamental result that the general case of any smooth maximizer will become indistinguishable from the case of a moral-expectation-maximizer (a "Bernoulli") or of a money-expectation-maximizer (a "Pascal") as the range of uncertainty of the outcomes shrinks toward zero. This has its good and bad implications. A bad implication is the fact that we cannot hope to distinguish between a "Bernoulli," a "Pascal," or a (smooth!) "Dostoevsky" by experiments that involve small dispersions (one evening's gamblings?).

A good implication is the fact that we can hope to determine a guinea pig's subjective or personal probabilities about particular contingent events by observing his betting behavior connected with such events when the prize and penalties associated with those events are "small." Ramsey and others have suggested this possible operational method for approximating to personal probabilities; and our present result demonstrates that this method of "revealed preference," generalized to become a method of "revealed personal probabilities," does not depend upon the assumption that the observed person maximizes the first moment of a utility function.[5]

[5] F. P. Ramsey, *op. cit.*, p. 176. As many scholars have observed, there is a dilemma in all small-prize experiments. If we make the prize small enough to minimize the changing marginal utilities (cardinal or ordinal), we may destroy the motivation of the guinea pig to reveal to us his true opinions. (He may be too uninterested to do the mental work to find his opinion; he may spite us; etc.) Indeed, operationally, who dares assert that his "opinion" exists if it is not in principle observable? This may be a social science analogue to the Heisenberg "uncertainty principle" in quantum physics. Just as we must throw light on a small object to see it and if it is small enough must thereby inevitably distort the object by our observational process, so we must to motivate a human guinea pig shower him with finite dispersion, whose effect may be to change his marginal utilities and contaminate his revealed probabilities. In principle, we can move to large scale dispersion experiments: thus, to determine whether Paul thinks rain more probable for July 4 than fair weather, let us threaten him with death if his guess is wrong and then observe which forecast he makes. Such "destructive tests" are expensive, even in these days of foundation philanthropy. Also, as Ramsey's p. 177 discussion of "ethically neutral" entities shows, we must be sure that the pig does not think that dying on a rainy day sends him to Paradise; for such a belief would violate the implicit independence assumption we make in separating out a man's probability beliefs from his evaluation of outcomes.

To prove (4), we shall show that (3) implies

$$(5)\quad \frac{\partial W(X, X, \cdots, X; p_1, p_2, \cdots, p_n)}{\partial X_i} = \frac{\partial W(X; 1)}{\partial X} p_i \quad (i = 1, 2, \cdots, n).$$

This important identity is most easily proved for rational p_i, but then by continuity considerations above can be extended to all real p_i. We suppose $p_i = m_i/\sum m_j = m_i/m$ and rewrite W in terms of m rather than n arguments,

$$(6)\quad W\left(X, X, \cdots, X; \frac{m_1}{m}, \cdots, \frac{m_n}{m}\right) = W\left(X, X, \cdots, X; \frac{1}{m}, \frac{1}{m}, \cdots, \frac{1}{m}\right)$$

where $m = m_1 + \cdots + m_n \geqq n$. By symmetry

$$\frac{\partial W\left(X, X, \cdots, X; \frac{1}{m}, \cdots, \frac{1}{m}\right)}{\partial X_j} = \frac{\partial W\left(X, X, \cdots, X; \frac{1}{m}, \cdots, \frac{1}{m}\right)}{\partial X_1}$$

$$(j = 2, \cdots, m)$$

and since changing one X_i on the left of (6) is like changing m_i X's on the right,

$$\frac{\partial W\left(X, \cdots, X; \frac{m_1}{m}, \cdots, \frac{m_n}{m}\right)}{\partial X_i} = m_i \frac{\partial W\left(X, \cdots, X; \frac{1}{m}, \cdots, \frac{1}{m}\right)}{\partial X}$$

$$= \frac{m_i}{m} \frac{\partial W(X; 1)}{\partial X}.$$

This proves (5) for all rational probabilities; and hence for all continuously differentiable W functions a similar relation holds for all real probabilities.

Relation (4) follows[6] directly from (5) if we use the Taylor's expansion

$$W(X_1, X_2, \cdots; p_1, p_2, \cdots) =$$

$$W[\overline{X} + (X_1 - \overline{X}), \overline{X} + (X_2 - \overline{X}), \cdots; p_1, p_2, \cdots]$$

$$= W(\overline{X}; 1) + \frac{\partial W(\overline{X}; 1)}{\partial X} \sum_1^n p_j (X_j - \overline{X}) + \cdots$$

[6] The above heuristic argument needs amplification. Specifically, if W is a functional of a probability distribution $P(X)$, where $\int_0^\infty dP(X) = 1$, then the existence of functional derivatives of order two of W at any point where $P(X)$ is varied should be shown to make W capable of being approximated in some sense by the linear expression $\alpha + \beta \int_0^\infty (X - A) dP(X)$ so long as $P(X)$ is in a close region of $H(X - A)$, where $H(t) = 1$ for $t > 0$, $H(t) = 0$ for $t \leqq 0$. The limits of the approximation should be rigorously established.

8. We can now prove the divergence of our double limit v/y, where $v = V(n, y; x)$ is defined by solving the implicit equation

$$W(x;1) = W(x - v + y2, x - v + y2^2, \cdots, x - v + y2^n, x; 2^{-1}, 2^{-2}, \cdots, 2^{-n}, 2^{-n}). \tag{7}$$

If $V(\infty, y; x)$ is infinite for all small y around x, our infinite limit for v/y is proved. If $V(\infty, y; x)$ is finite, since $V(\infty, 0; x) = 0$, the limit for v/y equals the limit for $\partial V/\partial y$. But

$$\frac{\partial V}{\partial y} = \frac{\sum_1^n 2^i \frac{\partial W}{\partial X_i}}{\sum_1^n \frac{\partial W}{\partial X_i}}.$$

It follows from (5) that as $y \to 0$ and all the X_i's coalesce to x,

$$\frac{\partial V}{\partial y} \to \frac{\sum_1^n 2^i 2^{-i}}{\sum_1^n 2^{-i}}.$$

Hence, $\lim_{n\to\infty} \lim_{y\to 0} \frac{\partial V}{\partial y} = \lim_{n\to\infty} \frac{n}{1 - 2^{-n}} = \infty$.

The order of these limits can be reversed by the same arguments used in the moral expectation case. Solow's Theorem—that for any probabilities $(p_1, \cdots, p_n)$, as the size of the stakes parameter y goes to zero, v/y equals the expected money gain $E(Z) = \sum p_i Z_i$, where $x - v + yZ_i$ represents the outcome of money price yZ_i occurring with probability p_i—also holds in the more general case.

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COMPETITIVE STABILITY UNDER WEAK GROSS SUBSTITUTABILITY: THE "EUCLIDEAN* DISTANCE" APPROACH*

BY KENNETH J. ARROW AND LEONID HURWICZ[1]

1. INTRODUCTION

1.1. IN EARLIER PAPERS [3], [1], the global stability of the competitive equilibrium was investigated under different assumptions on the excess demand functions. For the most part, the dynamic assumption was that the price of each commodity moved proportionately to its excess demand; there may or may not be a commodity distinguished as *numéraire* whose price is held fixed at 1.

Let there be $m + 1$ commodities, numbered $0, \cdots, m$; the *numéraire*, if any, is commodity 0. Let P_k be the price of commodity k, P the vector with components P_k $(k = 0, \cdots, m)$, and p the vector with components P_j $(j = 1, \cdots, m)$; the components of p are sometimes also denoted by p_j. All prices are non-negative, and at least one is positive. Let $F_k(P)$ be the excess demand for commodity k at price vector P; we assume,

(W) $$P \cdot F(P) = 0 ,$$

and for each $k = 0, 1, \cdots, m$:

(H) $F_k(P)$ is (positively) homogeneous of degree 0;

(C) $F_k(P)$ is continuous and differentiable;

(B) $F_k(P)$ is bounded from below.

Assumption (H) is usual and follows from utility maximization subject to the budget constraint.

The assumption (C) of continuity is to be understood here in an extended sense which permits positive infinite values of demand [negative infinite values are excluded by (B)]. If $F_j(P)$ is finite at some P, then continuity has the usual meaning; if infinite, then continuity means that

$$\lim_{n \to \infty} F_j(P^n) = + \infty ,$$

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for any sequence $\{P^n\}$ converging to P.[1] It may be remarked that, because of assumptions (W) and (B), $F_j(P)$ can only be infinite when $P_j = 0$. Admitting infinite excess demands for free goods does not seem unreasonable; if, however, the reader prefers to insist on continuity in the stricter sense which requires finite-valuedness, all the results of this paper will, of course, be *a fortiori* true.[2]

Assumption (W) is Walras' Law.

Assumption (B) is reasonable for the case of pure trade, in which case the maximum excess supply which an individual can offer is his initial holdings (see also [2], Section 2).

1.2. If (*a*) there is *no numéraire*, the dynamic system is

$$\dot{P}_k = \begin{cases} 0 & \text{if } P_k = 0,\ F_k(P) < 0\,, \\ F_k(P) & \text{otherwise } (k = 0, \cdots, m)\,. \end{cases} \tag{1}$$

If (*b*) there is *a numéraire*, let $f_j(p) = F_j(1, p)$ be the excess demand for commodity j as a function of $p_1, \cdots, p_m$, when $P_0 = 1$. The dynamic system in this case is

$$dp_j/dt = \begin{cases} 0 & \text{if } p_j = 0, f_j(p) < 0, \\ f_j(p) & (j = 1, \cdots, m)\,. \end{cases} \tag{2}$$

In both cases, the price of each commodity (other than the *numéraire*) varies proportionately to excess demand with an exception to prevent prices from becoming negative. By choice of suitable units of measurement we may assume that the rate of change of prices equals excess demand; i.e., there is no loss of generality in choosing all proportionality coefficients equal to unity.

The dynamic system (1) will be referred to as the *non-numéraire* system, (2) as the *numéraire* system (the less descriptive terms, "non-normalized," and "normalized," respectively, were used in [1], Section 2).

1.3. In [3] and [1] considerable attention was given to the case where the commodities are all gross substitutes, that is $\partial F_j/\partial P_k > 0$, for all $j \neq k$. It was shown that both the *numéraire* and *non-numéraire* systems were stable in the (rather strong) sense that, beginning with any starting point, the solution of the differential

[1] Cf. [3], Theorem 7 and footnote 36; also [2], Section 2.

[2] It should be noted that the excess demand function resulting from maximizing any utility function with the frequently-assumed property of positive marginal utilities $\left(\text{e.g., } \sum_{k=0}^{m} \alpha_k \log X_k,\ \alpha_k > 0\right)$ will tend to infinity as the corresponding price tends to zero; the function is continuous in the extended sense used here, but is not bounded as price varies over the unit simplex.

equations (1) or (2) converged to the (unique) equilibrium point (see [1], Section 4). Two methods of proof were given; in one, the "maximum norm" method, it was shown that the expression,

$$\max_k |(P_k/\overline{P}_k) - 1| ,$$

where $\overline{P}$ is any equilibrium point, was necessarily decreasing along any solution of (1) or (2) which did not start from an equilibrium point. In the second, the "Euclidean distance" approach, it was shown similarly that the square of the distance to any equilibrium,

$$\sum_{k=0}^{n} (P_k - \overline{P}_k)^2 ,$$

was decreasing along solutions of (1) and (2). The last fact follows from the statement that

(3) $\overline{P} \cdot F(P) > 0$, provided $\overline{P}$ is equilibrium and P is not.

Statement (3) can, as shown in [3], p. 534, be interpreted as saying that Samuelson's Weak Axiom of Revealed Preference holds between any pair of price vectors of which exactly one is an equilibrium price vector.[4]

In subsequent unpublished work, Uzawa [6], McKenzie [4], and Morishima [5], extended these results to the case where the commodities are *weak* gross substitutes, that is, where it is only assumed that

(S) $\begin{cases} \text{the derivatives } \partial F_j/\partial P_k \text{ exist for all } j, k, \\ \text{and } \partial F_j/\partial P_k \geq 0 \text{ for all } j \neq k . \end{cases}$

All three made assumptions which implied the existence of an equilibrium price vector with all components positive. In this note we remove these restrictions by an extension of the "Euclidean distance" argument. That is, we show that (S) in conjunction with the other assumptions is sufficient to imply stability even if the equilibrium points have some zero components.

1.4. In the case of weak gross substitutes the equilibrium is not necessarily unique. As a result, there is more than one conceivable meaning of stability. Both Uzawa and McKenzie demonstrate the following property, called by Uzawa "quasi-stability" ([7], pp. 3–4): every solution (path) of the differential equations (1) or (2) is bounded, and the distance from the moving point to the *set* of equilibria approaches zero. (Both Uzawa and McKenzie have dynamic systems more general than (1) and (2).) It does not follow from quasi-stability that

[4] This is, of course, considerably weaker than assuming that the Weak Axiom of Revealed Preference holds for *all* pairs of price vectors.

the solution approaches any limit; for instance, it might oscillate in some way about the set of equilibrium points. The "Euclidean distance" method, however, establishes that in fact each solution does converge to a limit, which must, of course, be an equilibrium. Thus we establish the stability of the system in the sense used in [3], p. 524.[6]

1.5. In this paper we confine ourselves to the dynamic systems (1) and (2). In [1], Theorem 1, it was shown that the "maximum norm" argument demonstrated the stability of more general dynamic systems, in which the rate of change of prices may be, for example, a non-linear function of excess demand. The results of Uzawa and McKenzie apply to these more general systems. It is not known whether or not their theorems generalize to the case where there does not exist at least one strictly positive equilibrium vector; it can be shown that if equilibria with some zero components are admitted, the solutions to the dynamic system which generalizes (1) can be unbounded.

1.6. Our procedure is based on previous results which showed that (3) implies both global stability and convexity of the set of equilibrium points (see [2], Theorem 2), provided solutions of (1) or (2) exist and are unique and continuous with respect to the starting point. Hence, we need only demonstrate that (3) holds. We first show that weak gross substitutability implies that the excess demand for a free good must be independent of the prices of all other goods. Second, we show that if there exists a positive equilibrium price vector, then (3) holds in the case of weak gross substitutes; the method is a modification of the argument used in [1] for the case of strong gross substitutes. From these two results we can derive the validity of (3) (see Theorem 1 below).

In the argument use is made several times of a reduction of the commodity space to a smaller number of dimensions in the following sense: if the prices of a set S of commodities are put equal to zero ($P_k = 0$ for $k \in S$), and if the excess demands for the free goods are non-positive, then the system of excess demand functions for the remaining commodities, $F_k(P)$ $(k \notin S)$, satisfies all the assumptions, (H), (C), (W), (B), and (S), considered as functions of the remaining prices, P_k $(k \notin S)$. Any results proved for such systems of functions can be applied to the reduced system (see Lemma 3).

2. EXCESS DEMAND FOR FREE GOODS

For any vector X and any set of indices S, we will denote by X_S

[6] Morishima's assumptions implied the uniqueness of equilibrium, so this question does not arise in his proof.

the vector with components X_j ($j \in S$); thus $P_S = \{P_{j_1}, \cdots, P_{j_s}\}$, where $j_1, \cdots, j_s$ are the elements of S. The notation $F_S(P)$ is understood similarly. Also, $\tilde{S}$ is the complement of S with respect to the set of indices $\{0, \cdots, m\}$.

We first demonstrate:

LEMMA 1. *If* (H), (C), (W), (B), *and* (S) *hold, then there are constants* F_j^0 *such that* $F_j(P) = F_j^0$ *for all* P *such that* $P_j = 0$. *Equivalently, under the assumptions listed, if* P' *and* P'' *are the price vectors such that* $P_j' = P_j'' = 0$, *then* $F_j(P') = F_j(P'')$.

PROOF: Let P be any fixed vector such that $P_j = 0$, k any fixed index, $k \neq j$. Write

$$S = \{r: P_r = 0\}, \text{ so that } P_S = 0 \text{ always,} \tag{4}$$

and

$$T = \{r: P_r > 0, r \neq k\} . \tag{5}$$

First, suppose $P_k > 0$, and consider two subcases, according as T is or is not null. In the first case,

$$F_j(P) = F_j(P_S, P_k) = F_j(0, P_k) = F_j(0, 1) ,$$

by (H), so that F_j is independent of P_k. In the second case, $F_j(P) = F_j(0, P_T, P_k)$. By homogeneity and Euler's equation,

$$\sum_{r \in T} (\partial F_j/\partial P_r)P_r + (\partial F_j/\partial P_k)P_k = 0 .$$

From (S), each term is non-negative, hence they must all be zero. In particular,

$$\partial F_j/\partial P_k = 0 .$$

Thus F_j is independent of P_k for all P_k positive and, by continuity (C) of $F_j(P)$, for $P_k = 0$; since this holds for all $k \neq j$, the lemma is proved.

In the case of strict gross substitutability ($\partial F_j/\partial P_k > 0$ for $j \neq k$), it was shown in [1], Lemma 1, that the excess demand for a free good is necessarily infinite, a special case of Lemma 1 of the present paper.

LEMMA 2. *If* $F_j(P^1) \leq 0$ *for some* P^1 *for which* $P_j^1 = 0$, *then* $F_j(P) \leq 0$ *for all* P.

PROOF: For any P, define Q so that

$$Q_j = 0, \quad Q_k = P_k \text{ for all } k \neq j . \tag{6}$$

From (H) and (S) it easily follows that $F_j(P)$ is a non-increasing function of P_j; from (6),

(7) $$F_j(P) \leq F_j(Q) .$$

From Lemma 1, $F_j(Q) = F_j(P^1) \leq 0$, by hypothesis; the lemma follows from (7).

LEMMA 3. *If assumptions* (H), (C), (W), (B), *and* (S) *hold for a vector* $F(P)$ *of excess demand functions, and if, for some set* S *of indices, the inequality* $F_{\tilde{s}}(P) \leq 0$ *holds for all* P, *then the vector of excess demand functions* $F_s(P_s, 0)$ *also satisfies* (H), (C), (W), (B), *and* (S) *as functions of* P_s; *further, if* $\overline{P}_s$ *is any equilibrium point for* $F_s(P_s, 0)$, *then* $(\overline{P}_s, 0)$ *is an equilibrium point for* $F(P)$.

PROOF: That assumptions (H), (C), (B), and (S) hold for $F_s(P_s, 0)$ is obvious. If the excess demand functions $F(P)$ are bounded from above as well as from below, (W) obviously holds also. However, we need to show that no difficulty will arise when $F(P)$ may be unbounded from above. Since (W) holds for $F(P)$, we can write

$$P \cdot F(P) = P_s \cdot F_s(P_s, P_{\tilde{s}}) + P_{\tilde{s}} \cdot F_{\tilde{s}}(P_s, P_{\tilde{s}}) = 0 \text{ for all } P .$$

If we let $P_{\tilde{s}}$ approach zero monotonically, it follows from (S) that F_s is monotone decreasing and therefore bounded by (B). At the same time $F_{\tilde{s}}$ is bounded from below by (B), and from above by the hypothesis $F_{\tilde{s}}(P) \leq 0$; hence, in the limit we have

$$P_s \cdot F_s(P_s, 0) = 0 ,$$

which is the assertion (W) for the set of functions $F_s(P_s, 0)$.

If $\overline{P}_s$ is an equilibrium for $F_s(P_s, 0)$, then by definition

$$F_s(\overline{P}_s, 0) \leq 0 .$$

Since $F_{\tilde{s}}(\overline{P}_s, 0) \leq 0$ by hypothesis, $F(\overline{P}_s, 0) \leq 0$, so that $(\overline{P}_s, 0)$ is an equilibrium by definition.

3. THE "REVEALED PREFERENCE" RELATION BETWEEN EQUILIBRIUM AND DISEQUILIBRIUM POINTS

THEOREM 1. *If* $F(P)$ *is a vector of excess demand functions satisfying* (H), (C), (W), (B), *and* (S), *then*

$$\overline{P} \cdot F(P) > 0$$

provided $\overline{P}$ *is an equilibrium price vector and* P *is not* $(P \neq 0)$.

In Section 3.1 we prove the theorem for the case $\overline{P} > 0$; in 3.2 the general case is established.

3.1. The proof follows in general the lines of [1], Lemma 5. However, some modifications are needed: (1) In [1] the proof made essential use of the assumption of gross substitutability in the strict sense; as

we shall see, this assumption is not needed.[6] In [1] it was assumed that the disequilibrium price vector was positive; again we shall find this assumption unnecessary.

3.1.1. The magnitude $\overline{P} \cdot F(P)$ is measured in money and is independent of the units in which commodities are measured. Since (in 3.1) $\overline{P} > 0$, we may, without loss of generality, assume that

$$\overline{P}_k = 1 \qquad (k = 0, \cdots, m), \tag{8'}$$

so that

$$F(\overline{P}) = 0 . \tag{8''}$$

For a given non-zero vector $P \equiv (P_0, P_1, \cdots, P_m)$ assume the commodities so numbered that

$$P_k \geq P_{k+1} \qquad (k = 0, \cdots, m-1), \tag{9'}$$

and hence, by $P \neq 0$,

$$P_0 > 0 . \tag{9''}$$

Then define a sequence of vectors $P^s (s = 0, \cdots, m)$ by the conditions that

$$P_k^s = \max (P_s, P_k) . \tag{10}$$

It may easily be seen from (9') and (10) that for $P^s = (P_0^s, P_1^s, \cdots, P_m^s)$

$$P_k^s = \begin{cases} P_s & (k \leq s), \\ P_k & (k > s), \end{cases} \tag{11}$$

and

$$P_k^{s+1} = \begin{cases} P_k & (k \leq s), \\ P_{s+1} & (k > s). \end{cases} \tag{12}$$

Thus the change from P^s to P^{s+1} consists in changing the last $m - s$ components from P_s to P_{s+1}. In more detail, we see that

$$\begin{aligned} P^0 &= (P_0, \cdots, P_0) , \\ P^1 &= (P_0, P_1, P_1, \cdots, P_1) , \\ &\cdots \\ P^s &= (P_0, \cdots, P_{s-1}, P_s, P_s, \cdots, P_s) , \\ P^{s+1} &= (P_0, \cdots, P_{s-1}, P_s, P_{s+1}, \cdots, P_{s+1}) , \\ &\cdots \\ P^m &= (P_0, \cdots, P_{m-1}, P_m) . \end{aligned}$$

[6] Morishima [5] replaced the assumption of strict gross substitutability by one of indecomposability, but this is also unnecessary.

From (8′) we see that $P^0 = P_0\bar{P}$; it follows from (H), (9″), and (8″) that

$$F(P^0) = 0 . \tag{13}$$

Also, we note that

$$P^m = P . \tag{14}$$

The last $m - s$ components of P^{s+1} are, from (11), (12), and (9′), not greater than the corresponding components of P^s, while the first $s + 1$ components are the same. Hence, by substitutability (S)

$$F_k(P^{s+1}) \leq F_k(P^s) \qquad (k \leq s) . \tag{15}$$

Suppose for the moment that $P_{s+1} > 0$; then from (9′) $P_s > 0$. Write $Q^{s+1} = (P_s/P_{s+1})P^{s+1}$. Since by (9′) $P_s/P_{s+1} \geq 1$, we have $Q_k^{s+1} \geq P_k^s$ $(k \leq s)$, $Q_k^{s+1} = P_k^s$ $(k > s)$ from (11) and (12). By (S) $F_k(Q^{s+1}) \geq F_k(P^s)$ $(k > s)$; since by (H) $F_k(Q^{s+1}) = F_k(P^{s+1})$, we have

$$F_k(P^{s+1}) \geq F_k(P^s) \qquad (k > s) . \tag{16}$$

The assumption that $P_{s+1} > 0$ may now be dropped, since by continuity (C) (16) also holds if $P_{s+1} = 0$.

The inequality (16) can also be written in the form

$$F_k(P^{s+1}) \geq F_k(P^t) \qquad (k > s \geq t \geq 0) .$$

By induction $F_k(P^{s+1}) \geq F_k(P^0)$, $(k > s \geq t \geq 0)$. But the right member of the last inequality vanishes by (13), and if we set t equal to $s - 1$, we get

$$F_k(P^s) \geq 0 \qquad (k > s) . \tag{17}$$

3.1.2. In 3.1.3 below we shall use the results of 3.1.1 to establish the following inequalities:

$$\sum_{k=0}^{m} F_k(P^{s+1}) \geq \sum_{k=0}^{m} F_k(P^s) \quad (s = 0, 1, \cdots, m-1) , \tag{18}$$

$$\sum_{k=0}^{m} F_k(P^{s+1}) > \sum_{k=0}^{m} F_k(P^s) \quad \begin{cases} \text{if } P^s \text{ is an equilibrium vector,} \\ P^{s+1} \text{ a disequilibrium vector,} \\ s = 0, 1, \cdots, m-1. \end{cases} \tag{19}$$

From (18) it follows by induction that $\sum_{k=0}^{m} F_k(P^s) \geq \sum_{k=0}^{m} F_k(P^0)$ for all s. Set $s = m$ and recall (13), then

$$\sum_{k=0}^{m} F_k(P^m) \geq 0 .$$

From (8′) and (14) we can write

$$\overline{P} \cdot F(P) \geq 0 \text{ for all } P . \tag{20}$$

If, in addition, we assume that $P = P^m$ is a disequilibrium vector, then, since P^0 is an equilibrium vector, there must be some s for which P^s is an equilibrium vector, P^{s+1} a disequilibrium vector. Then with the aid of (19) we have by the same argument

$$\overline{P} \cdot F(P) > 0 \text{ for any disequilibrium } P , \tag{21}$$

which is the assertion of the theorem.

3.1.3. To prove (18) and (19), we consider *three cases*: (*a*) $P_s = P_{s+1}$; (*b*) $P_s > P_{s+1}$ and $F_{k'}(P^{s+1}) < F_{k'}(P^s)$ for some $k' \leq s$; (*c*) $P_s > P_{s+1}$ and $F_k(P^{s+1}) = F_k(P^s)$ for all $k \leq s$. (In view of (9) and (15), these cases are exhaustive.)

(*a*) In this case, from (10) $P^s = P^{s+1}$; hence

$$\sum_{k=0}^{m} F_k(P^{s+1}) = \sum_{k=0}^{m} F_k(P^s) \text{ if } P_s = P_{s+1} . \tag{22}$$

(*b*) In this case we have

$$P_{k'}[F_{k'}(P^{s+1}) - F_{k'}(P^s)] < P_{s+1}[F_{k'}(P^{s+1}) - F_{k'}(P^s)] , \tag{23}$$

since $P_{k'} \geq P_s > P_{s+1}$ by the assumption that $k' \leq s$ and (9). From (15)

$$P_k[F_k(P^{s+1}) - F_k(P^s)] \leq P_{s+1}[F_k(P^{s+1}) - F_k(P^s)] \qquad (k \leq s) .$$

This, in conjunction with (23), yields

$$\sum_{k=0}^{s} P_k[F_k(P^{s+1}) - F_k(P^s)] < P_{s+1} \sum_{k=0}^{s} [F_k(P^{s+1}) - F_k(P^s)] . \tag{24}$$

Now by (W) we have

$$\sum_{k=0}^{m} P_k^s F_k(P^s) = 0 = \sum_{k=0}^{m} P_k^{s+1} F_k(P^{s+1}) ,$$

and therefore, with the aid of (11) and (12) and then (24),

$$\begin{aligned} 0 = \sum_{k=0}^{m} P_k^{s+1} F_k(P^{s+1}) - \sum_{k=0}^{m} P_k^s F_k(P^s) &= \sum_{k=0}^{s} P_k[F_k(P^{s+1}) - F_k(P^s)] \\ &+ P_{s+1} \sum_{k=s+1}^{m} [F_k(P^{s+1}) - F_k(P^s)] + (P_{s+1} - P_s) \sum_{k=s+1}^{m} F_k(P^s) \\ &< P_{s+1} \sum_{k=0}^{m} [F_k(P^{s+1}) - F_k(P^s)] + (P_{s+1} - P_s) \sum_{k=s+1}^{m} F_k(P^s) \\ &\leq P_{s+1} \sum_{k=0}^{m} [F_k(P^{s+1}) - F_k(P^s)] , \end{aligned} \tag{25}$$

the last inequality following from (17) and the fact that $P_{s+1} - P_s < 0$. From (25) we must have $P_{s+1} > 0$, and therefore (25) yields

$$(26)\quad \sum_{k=0}^{m} F_k(P^{s+1}) > \sum_{k=0}^{m} F_k(P^s) \quad \begin{cases} \text{if } P_{s+1} < P_s \text{ and} \\ F_{k'}(P^{s+1}) < F_{k'}(P^s) \text{ for some } k' \leq s\,. \end{cases}$$

(c) In this case by assumption

$$\sum_{k=0}^{s} F_k(P^s) = \sum_{k=0}^{s} F_k(P^{s+1})\,,$$

so that from (16)

$$(27)\quad \sum_{k=0}^{m} F_k(P^{s+1}) \geq \sum_{k=0}^{m} F_k(P^s) \quad \begin{cases} \text{if } P_s > P_{s+1} \text{ and} \\ F_k(P^{s+1}) = F_k(P^s) \text{ for all } k \leq s\,, \end{cases}$$

which are the assumptions of this case.

Suppose now that P^s is an equilibrium, P^{s+1} a disequilibrium vector. Then $F(P^s) \leq 0$, so that from (15) $F_k(P^{s+1}) \leq 0$ $(k \leq s)$. Since P^{s+1} is not an equilibrium vector, $F_k(P^{s+1}) > 0$ for some k so that

$$F_{k''}(P^{s+1}) > 0 \text{ for some } k'' > s\,.$$

Since $F_{k''}(P^s) \leq 0$, $F_{k''}(P^{s+1}) > F_{k''}(P^s)$, therefore, because of (16),

$$(28)\quad \sum_{k=0}^{m} F_k(P^{s+1}) > \sum_{k=0}^{m} F_k(P^s) \begin{cases} \text{if } P_s > P_{s+1} \text{ while} \\ F_k(P^{s+1}) = F_k(P^s) \text{ for all } k \leq s, \text{ with} \\ P^s \text{ equilibrium, } P^{s+1} \text{ disequilibrium.} \end{cases}$$

Then (18) follows from (22), (26), and (27). If P^s is equilibrium and P^{s+1} is disequilibrium, then $P^s \neq P^{s+1}$ and therefore $P_s > P_{s+1}$; then (19) follows from (26) or (28). Hence, by the argument of Section 3.1.2, (20) and (21) are demonstrated.

3.1.4. *Remark.* Let P be any equilibrium point; then $F(P) \leq 0$. From (20) it follows that

$$(29)\quad \begin{cases} \text{if there exists a positive equilibrium, then } F(P) = 0 \\ \text{for every equilibrium price vector } P. \end{cases}$$

3.2. We now consider the general case where the equilibrium $\overline{P}$ may have zero components. Write

$$(30)\qquad S = \{k: \overline{P}_k > 0\}\,.$$

Then $\overline{P}_{\bar{s}} = 0$. By definition of equilibrium $F(\overline{P}) \leq 0$ and, in particular, $F_{\bar{s}}(\overline{P}) \leq 0$; by Lemma 2,

$$(31)\qquad F_{\bar{s}}(P) \leq 0 \quad \text{for all } P\,.$$

Then from (B), $F_{\tilde{s}}(P)$ is finite-valued, so that the product $\overline{P}_{\tilde{s}} \cdot F_{\tilde{s}}(P)$ is well defined and equal to zero. Hence,

$$\overline{P} \cdot F(P) = \overline{P}_s \cdot F_s(P) + \overline{P}_{\tilde{s}} \cdot F_{\tilde{s}}(P) = \overline{P}_s \cdot F_s(P) . \tag{32}$$

Since $P_{\tilde{s}} \geq 0$, it follows from (S) that

$$F_s(P) = F_s(P_s, P_{\tilde{s}}) \geq F_s(P_s, 0) , \tag{33}$$

so that

$$\overline{P}_s \cdot F_s(P) \geq \overline{P}_s \cdot F_s(P_s, 0) . \tag{34}$$

From (31) and Lemma 3 the function $F_s(P_S, 0)$ satisfies all the assumptions (H), (C), (W), (B), and (S). Further, it has a positive equilibrium $\overline{P}_S$. Hence, we can apply the results (21) and (29) of Section 3.1 and conclude that

$$\overline{P}_s \cdot F_s(P_s, 0) > 0 \text{ if } P_s \text{ is not an equilibrium of } F_s(P_s, 0) , \tag{35}$$

$$F_s(P_s, 0) = 0 \text{ if } P_s \text{ is an equilibrium of } F_s(P_s, 0). \tag{36}$$

If P_s is not an equilibrium of $F_s(P_s, 0)$, then $\overline{P} \cdot F(P) > 0$ from (32), (34-35). If it is, then $F_s(P) \geq 0$ from (33) and (36). If $F_s(P) = 0$, then, from (31), P is an equilibrium. Hence, if P is a disequilibrium vector, $F_s(P) \geq 0$, $F_s(P) \neq 0$, so that

$$\overline{P}_s \cdot F_s(P) > 0 ,$$

and, by (32), $\overline{P} \cdot F(P) > 0$ even if P_s is an equilibrium of $F_s(P_s, 0)$.

4. STATEMENT OF THEOREM ON STABILITY AND CONVEXITY OF EQUILIBRIA

In [1], as part of the proof of Theorem 2, it was shown that the condition $\overline{P} \cdot F(P) > 0$ was sufficient for the convergence of solutions of both the *non-numéraire* and *numéraire* systems; for further discussion see [2], Theorem 2, where "corners" are explicitly treated, and it is also shown that the same condition insures the set of equilibria is convex. Hence, Theorem 1 implies

THEOREM 2. *If $F(P)$ is a vector of excess demand functions which are homogeneous of degree zero, continuous, bounded from below, if Walras' Law holds, and if all commodities are weak gross substitutes, then both the non-numéraire and numéraire adjustment systems (Equations* (1) *and* (2) *above) are stable*[7] *and the set of equilibria is convex.*[8]

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[7] In the sense that every solution (path) converges to some equilibrium point.

[8] The convexity of the set of equilibria when all commodities are weak gross substitutes was observed by McKenzie [4].

REFERENCES

[1] ARROW, KENNETH, J., HENRY D. BLOCK, AND LEONID HURWICZ, "On the Stability of the Competitive Equilibrium, II," *Econometrica*, Vol. 27 (1959), pp. 82-109.

[2] ARROW, KENNETH J., AND LEONID HURWICZ, "Some Remarks on the Equilibria of Economic Systems," Technical Report No. 76, O. N. R. Contract Nonr-225 (50) (NR-047-004) Stanford University, July 23, 1959.

[3] ———, "On the Stability of the Competitive Equilibrium, I," *Econometrica*, Vol. 26 (1958), pp. 522-552.

[4] MCKENZIE, LIONEL, "A Contribution to the Theory of Stability of Competitive Equilibrium," unpublished manuscript, 1959.

[5] MORISHIMA, MICHIO, "Gross Substitutability, Homogeneity, and the Walras Law," unpublished manuscript, 1959.

[6] UZAWA, HIROFUMI, "An Alternative Proof of the Stability in the Gross Substitute Case," Technical Report No. 60, Office of Naval Research, Contract N6onr-25133, Stanford University, October 8, 1958.

[7] ———, "On the Stability of Dynamic Processes," Technical Report No. 61, Office of Naval Research, Contract N6onr-25133, Stanford University, November 28, 1958.

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STABILITY AND NON-NEGATIVITY IN A WALRASIAN TÂTONNEMENT PROCESS*

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1. INTRODUCTION

SINCE SOME RECENT STUDIES [1, 5, 7, 8, 11] thoroughly explored the static aspects of market mechanism by establishing the existence of competitive equilibrium, we have seen a revival of interest in the study of its dynamic aspects. One problem of major importance centers around the derivation of the (mainly global) stability of competitive equilibrium from several conditions of economic significance imposed upon excess demand functions. Important results in this line were recently obtained by Arrow and Hurwicz [2], Arrow, Block and Hurwicz [3], Hahn [6], McKenzie [9] and Negishi [10]. However, except for the Arrow-Block-Hurwicz article and the result of McKenzie, little explicit attention seems to have been paid to the non-negativity of prices in these studies. As was already noted in the early thirties, negative prices are unjustifiable from the economic viewpoint, and the theory of general equilibrium should explicitly take into account the non-negativity of economic variables. This restriction led Wald [16], and, in more recent times, several other authors [1, 5, 7, 8, 11] to investigate the existence of competitive equilibrium. The same restriction should also be relevant to the study of the dynamic aspects of a competitive market.

Working along this line, Nikaidô [12] attempted to offer some results on global stability with the non-negativity condition for a special type of competitive market described by the Brown-von Neumann differential equation. Uzawa [13, 14, 15] also proposed and studied a non-negative price adjustment process which is formulated in terms of a difference equation and is in complete agreement with the classical law of competition. This process, as he proved, enjoys global stability in an approximate way, although it may fail to be stable in the ordinary sense. In this note we are concerned with a further study of this process, and ordinary non-negative global stability will be proved for a differential equation version of the process in which

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the weak axiom of revealed preference holds between any equilibrium price vector and any other price vector not proportional to the former.

2. SUMMARY OF THE RESULTS

2.1. There are $n+1$ goods denoted by $i = 0, 1, 2, \cdots, n$; their *non-normalized* prices are denoted by $q_0, q_1, \cdots, q_n$. Let $q = (q_0, q_1, \cdots, q_n) \geqq 0, \neq 0$ stand for the price vector.[3]

$E_i(q) = E_i(q_0, q_1, \cdots, q_n)$, the excess demand for the ith good, is assumed to fulfill the following conditions:

(*a*) *$E_i(q)$ is single-valued and continuous for $q \geqq 0, \neq 0$.*

(*b*) $\sum_{i=0}^{n} q_i E_i(q) = 0$; *i.e., Walras' law holds true.*

The recent studies [1, 5, 7, 8, 11] have established, under assumptions (*a*) and (*b*), the existence of a competitive equilibrium price vector $\hat{q}$ such that

(i) $\quad E_i(\hat{q}) \leqq 0 \quad (i = 0, 1, \cdots, n)$

(ii) $\quad E_i(\hat{q}) = 0 \quad$ if $\hat{q}_i > 0$.

2.2. In the discussion below the following assumptions will also be made:

(*c*) *$E_i(q)$ is homogeneous of degree 0.*

(*d*) *The weak axiom of revealed preference holds in the sense that for any equilibrium price vector $\hat{q}$ and any other price vector q not proportional to $\hat{q}$ the inequality $\sum_{i=0}^{n} \hat{q}_i E_i(q) > 0$ prevails.*

Assumption (*d*) immediately entails the uniqueness of the equilibrium price vector up to multiplication by positive scalars.

2.3. We take up one good, say the 0-th good, to play the role of *numéraire*. Let

$$p = (p_1, p_2, \cdots, p_n), \quad p_i = q_i/q_0 \quad (i = 1, 2, \cdots, n)$$
$$F_i(p) = E_i(1, p_1, p_2, \cdots, p_n) \qquad (i = 0, 1, 2, \cdots, n),$$

so that the letter p is used exclusively to designate *normalized* price vectors. Uzawa [13, 14, 15] proposed the following iterative process corresponding to the Walrasian tâtonnement; that is,

$$p_i(t+1) = \max [\rho F_i(p(t)) + p_i(t), 0], \quad (i = 1, 2, \cdots, n),$$

where ρ is a positive constant. This system is obtained from the classical formulation of price adjustment

$$p_i(t+1) = \rho F_i(p(t)) + p_i(t), \ (i = 1, 2, \cdots, n)$$

by taking into account the downward rigidity of prices below zero.

[3] In what follows, for a vector q, the inequality $q \geqq 0$ means as usual that all the components of q are non-negative.

He proved that given any small ε neighborhood of the equilibrium price vector $\hat{p}$ and any initial position $p(0) \geqq 0$, the solution $p(t)$ of the above difference equation approaches and eventually reaches this prescribed neighborhood of $\hat{p}$ for small $\rho > 0$ depending on ε and $p(0)$, while the distance $p(t)$ from $\hat{p}$ monotonically decreases. The proof was done under assumptions (a), (*b*), and (*d*).

2.4. A differential equation analogous to the above equation is

$$\frac{dp_i}{dt} = \theta_i(p) - p_i\,, \quad (i = 1, 2, \cdots, n)\,, \tag{1}$$

where

$$\theta_i(p) = \max\left[\rho F_i(p) + p_i, 0\right]. \tag{2}$$

In the process formulated by the difference equation stated above, the increments of prices are given by

$$\begin{aligned}\Delta p_i(t) &= p_i(t+1) - p_i(t) \\ &= \max\left[\rho F_i(p(t)) + p_i(t), 0\right] - p_i(t)\,, \\ &\qquad (i = 1, 2, \cdots, n)\,.\end{aligned}$$

This suggests the replacement of Δp_i by dp_i/dt to get a differential equation version (1) of the process proposed by Uzawa. We are here concerned with the study of equation (1).

The main results of this note are summarized as follows:

PROPOSITION 1 (EXISTENCE). *Under assumptions* (*a*) *and* (*c*) *equation* (1) *has one solution* (*tâtonnement path*), *starting from any given initial position* $p(0) \geqq 0$.

PROPOSITION 2 (STABILITY). *If the price of each good is positive at equilibrium, i.e.,* $\hat{q}_i > 0$ $(i = 0, 1, \cdots, n)$, *then under assumptions* (*a*), (*b*), (*c*) *and* (*d*) *any solution* $p(t)$ *of* (1) *converges to* $\hat{p}$ *for small* $\rho > 0$. *During the convergence of* $p(t)$ *to* $\hat{p}$, *the distance of* $p(t)$ *from* $\hat{p}$ *monotonically shrinks to zero.*

2.5. Equation (1) avoids the various difficulties in the classical formulation of price adjustment in connection with the non-negativity of prices; it nevertheless retains the orthodox version of price flexibility in the sense that excess demand causes a rise in price, and excess supply entails a fall in price.[3] First, if $p_i = 0$, then $dp_i/dt = \theta_i(p) \geqq 0$, so that prices are never negative. We next check on the pattern of price flexibility. In fact, whenever $\rho F_i(p) + p_i \geqq 0$, the ith

[3] Here, the orthodox version of price flexibility is understood in such a broad sense that the rate of price change has the same sign as excess demand, but not necessarily depends on the amount of excess demand. See also the final section of this paper.

equation in (1) reduces to $dp_i/dt = \rho F_i(p)$. Furthermore, if $\rho F_i(p) + p_i < 0$, then $dp_i/dt = -p_i$, because $\theta_i(p) = 0$, and p_i decreases so long as $p_i > 0$, while satisfying $F_i(p) < -p_i/\rho \leqq 0$.

The above consideration also makes it clear that if $\rho F_i(p) < -p_i$, the orthodox system $dp_i/dt = \rho F_i(p)$ is replaced by another one, namely, $dp_i/dt = -p_i$. Furthermore, the switch from one system to the other occurs not suddenly but always gradually and continuously. This built-in shock absorber prevents the changing price $p(t)$ from crashing against the bottom $p_i = 0$ and from penetrating into the negative domain $p_i < 0$. It slows down the price movement near the bottom and always insures a gentle sidewise crawling. This reminds us of a flap device which works when a jet airliner does a soft landing.

2.6. Introducing coefficients $k_i > 0$ of speed adjustment into (1) would cause no difficulties, and the method and results in this note equally apply to the equation

$$k_i \frac{dp_i}{dt} = \theta_i(p) - p_i \quad (i = 1, 2, \cdots, n),$$

either by reducing it to equation (1) through the change of units, or by using some suitably weighted distance function.

3. PROOF OF PROPOSITION 1 (EXISTENCE)

3.1. We begin by proving three lemmas.

LEMMA 1. *Under assumptions (a) and (c) each $E_i(q)$ is bounded for $q \geqq 0, \neq 0$; this in particular implies that $F_i(p)$ is bounded for $p \geqq 0$.*

PROOF. Let $S_{n+1} = \{q \mid q_i \geqq 0, \sum_{i=0}^{n} q_i = 1\}$. $E_i(q)$ is continuous on S_{n+1} which is compact. Hence $E_i(q)$ is bounded on S_{n+1}, that is, $|E_i(q)| \leqq M_i$ on S_{n+1}. Then, by (c), $|E_i(q)| \leqq M_i$ for all $q \geqq 0, \neq 0$.

LEMMA 2. *The functions $\theta_i(p) - p_i$ appearing in (1) are bounded for $p \geqq 0$.*

PROOF. By Lemma 1, $|F_i(p)| \leqq N \equiv \max(M_0, M_1, \cdots, M_n)$ for all $p \geqq 0$. First, it is readily seen, in view of definition (2), that

$$|\theta_i(p) - p_i| \leqq \max[\rho F_i(p), p_i],$$

because $\theta_i(p) - p_i$ equals either $\rho F_i(p)$ or $-p_i$.

Now, if $0 \leqq p_i \leqq \rho N$, the above inequality immediately implies that $|\theta_i(p) - p_i| \leqq \rho N$. If, on the other hand, $\rho N < p_i$, then $\rho F_i(p) + p_i$ must be positive because $|\rho F_i(p)| \leqq \rho N$. Hence, in this case, $\theta_i(p) - p$ actually equals $\rho F_i(p)$, so that $|\theta_i(p) - p_i| \leqq \rho N$. Therefore, always

$$|\theta_i(p) - p_i| \leqq \rho N.$$

LEMMA 3. *Given any initial position* $p(0) = p^0 \geqq 0$, *equation* (1) *has a solution in a small time interval* $[0, \sigma]$, $\sigma > 0$, *starting from* p^0.

PROOF. Let $|p| = (|p_1|, |p_2|, \cdots, |p_n|)$. The equation

$$(3) \qquad \frac{dp_i}{dt} = \max[\rho F_i(|p|) + p_i, 0] - p_i \quad (i = 1, 2, \cdots, n)$$

has a solution in an interval $[0, \sigma]$, $\sigma > 0$, starting from p^0, by the Cauchy-Peano theorem,[4] since the right-hand side of (3) is continuous. If $p_i \leqq 0$, then $\max[\rho F_i(|p|) + p_i, 0] - p_i \geqq 0$ so that $p_i(t)$ never becomes negative. This can easily be rigorously proved, though it will not be given here. Therefore this solution $p(t)$ remains non-negative in $[0, \sigma]$ and satisfies equation (1).

3.2. PROOF OF PROPOSITION 1. Let ω be the least upper bound of σ, such that the solution $p(t)$ of Lemma 2 can be continued to $[0, \sigma]$. Suppose that $\omega < +\infty$. It will be shown that this supposition leads to a contradiction. By definition, there is a solution $p(t)$ in $[0, \omega)$, and for any $0 \leqq t_1 < t_2 < \omega$,

$$|p_i(t_2) - p_i(t_1)| \leqq \int_{t_1}^{t_2} |\theta_i(p(s)) - p_i(s)|\, ds \leqq \Gamma |t_1 - t_2|,$$

where Γ is a positive constant. Hence, $p_i(t_1) - p_i(t_2) \to 0$, as t_1 and t_2 tend to 0. This implies that

$$\lim_{t \to \omega} p_i(t) = p_i^\omega \qquad (i = 1, 2, \cdots, n)$$

exist, and $p(t)$ can be continued to $[0, \omega]$ by setting $p(\omega) = p^\omega$. By Lemma 3, there is a solution $\bar{p}(t)$ of (1) starting from $p(\omega)$, in some $[0, \tau]$, $\tau > 0$; $p(t)$ can therefore be continued beyond ω to $[0, \omega + \tau]$, by defining $p(t) = \bar{p}(t - \omega)$ in $[\omega, \omega + \tau]$, which yields a contradiction. Accordingly, $\omega = +\infty$, and $p(t)$ can be continued to $[0, +\infty)$.

4. PROOF OF PROPOSITION 2 (STABILITY)

Let us now take two sets X and Y; the symbol $X \backslash Y$ will stand for the difference set, i.e., $X \backslash Y = \{x \mid x \in X, x \notin Y\}$.

The proof will be carried out in the following four steps, (α) through (δ).

(α) By assumption, the unique normalized equilibrium price vector $\hat{p}$ is positive, i.e., $\hat{p}_i > 0$ $(i = 1, 2, \cdots, n)$, so that by condition (ii)

$$F_i(\hat{p}) = 0 \qquad (i = 0, 1, 2, \cdots, n),$$

and hence $F_i(\hat{p}) + \hat{p}_i > 0$ $(i = 1, 2, \cdots, n)$. Since the $F_i(p)$'s are continuous

[4] See, e.g., [4]. The uniqueness of the solution is not insured here.

for $p \geqq 0$, given any constant ρ with $0 < \rho \leqq 1$, there is a neighborhood[5,6] $U(\hat{p}, \varepsilon) \subset R^n_+$ such that

$$(4) \qquad \rho F_i(p) + p_i > 0 \quad (i = 1, 2, \cdots, n) \quad \text{for } p \in U(\hat{p}, \varepsilon) .$$

(β) By assumption (d), if $p \neq \hat{p}$,

$$f(p) \equiv F_0(p) + \sum_{i=1}^{n} \hat{p}_i F_i(p) > 0 .$$

We next prove

LEMMA 4. $\inf_{p \notin U(\hat{p}, \varepsilon)} f(p) = M > 0.$

PROOF. Clearly $M \geqq 0$. Suppose that $M = 0$. Then it entails the existence of a sequence $\{p^\nu\}$ in $R^n_+ \backslash U(\hat{p}, \varepsilon)$ such that $\lim_{\nu \to \infty} f(p^\nu) = M$. Define the continuous mapping $\varphi: R^n_+ \to S_{n+1}$ by the formulae

$$\varphi_0(p) = \frac{1}{1 + \sum_{i=1}^{n} p_i} , \varphi_i(p) = \frac{p_i}{1 + \sum_{i=1}^{n} p_i} \quad (i = 1, 2, \cdots, n) .$$

Because of the compactness of S_{n+1}, $\{\varphi(p^\nu)\}$ itself may be assumed[7] to converge to a q in S_{n+1}. By (c), we have

$$\varphi_0(\hat{p}) f(p) = \sum_{i=0}^{n} \varphi_i(\hat{p}) E_i(\varphi(p)) ,$$

so that, by continuity,

$$\sum_{i=0}^{n} \varphi_i(\hat{p}) E_i(q) = \lim_{\nu \to \infty} \varphi_0(\hat{p}) f(p^\nu) = \varphi_0(\hat{p}) M = 0 .$$

Therefore, by (d), q is an equilibrium price vector, and q must equal $\varphi(\hat{p})$, since both of them lie in S_{n+1}. This, in particular, implies that $q_i > 0 \, (i = 0, 1, \cdots, n)$, especially $q_0 > 0$. Since

$$\lim_{\nu \to \infty} \varphi_0(p^\nu) = \lim_{\nu \to \infty} \frac{1}{1 + \sum p_i^\nu} = q_0 > 0 ,$$

$1 + \sum_{i=1}^{n} p_i^\nu$ converges to $1/q_0$, that is, $\{p^\nu\}$ is a bounded sequence in R^n_+, and therefore itself may be assumed[8] to converge to a p. But the closed set $\Delta = R^n_+ \backslash U(\hat{p}, \varepsilon)$ contains p, and the limit p is contained in Δ. Then, by continuity,

[5] $U(\hat{p}, \varepsilon)$ is the set of vectors p whose distances to $\hat{p}$ are less than ε.

[6] R^n_+ denotes the so-called non-negative orthant, namely the totality of all nonnegative vectors of the n-dimensional space R^n.

[7] This can be done by passing to a suitable subsequence of $\{\varphi(p^\nu)\}$.

[8] This was done on a similar ground as remarked in footnote 7.

$$f(p) = \lim_{\nu\to\infty} f(p^\nu) = M \neq 0 ,$$

from which it follows, by (d), $p = \hat{p}$. This implies a contradiction $\hat{p} \in \Delta$. Hence $M > 0$.

(γ) By Lemma 1, $F_i(p)$'s $(i = 1, 2, \cdots, n)$ are bounded for $p \geqq 0$. Therefore there is $L > 0$ such that

$$\sum_{i=1}^{n} F_i(p)^2 \leqq L .$$

(δ) We now prove that for any constant ρ fulfilling

$$0 < \rho < \min(2M/L, 1) , \tag{5}$$

any solution $p(t)$ of (1) starting from any initial position $p(0) \geqq 0$ converges to $\hat{p}$. Step (δ) will be divided into the two portions (I) and (II).

(I) If $p(t) \in U(\hat{p}, \varepsilon)$, then, by (4) and $\rho < 1$, system (1) reduces to the classical system

$$\frac{dp_i}{dt} = \rho F_i(p) \quad (i = 1, 2, \cdots, n) .$$

Then, as is shown, e.g., in section 4.2 of the Arrow-Block-Hurwicz article [3], if, following Liapounov, we define $2V = \sum_{i=1}^{n} (p_i - \hat{p}_i)^2$, we have

$$\frac{dV}{dt} = \rho \sum_{i=1}^{n} (p_i - \hat{p}_i) F_i(p)$$
$$= -\rho f(p) \qquad \text{by } (b) .$$

(II) Now, by definition,

$$\theta_i(p) = \rho F_i(p) + p_i \qquad \text{if } \rho F_i(p) + p_i \geqq 0 .$$

and

$$\theta_i(p) = 0 > \rho F_i(p) + p_i \qquad \text{if } \rho F_i(p) + p_i < 0 ,$$

from which it follows that $\theta_i(p)^2 \leqq |\rho F_i(p) + p_i|^2$. Thus, substituting $p_i + \dot{p}_i$ for $\theta_i(p)$ by (1), we have

$$(p_i + \dot{p}_i)^2 \leqq (\rho F_i(p) + p_i)^2 = \rho^2 F_i(p)^2 + 2\rho p_i F_i(p) + p_i^2 .$$

Hence

$$\sum_{i=1}^{n} p_i^2 + 2\sum_{i=1}^{n} p_i \dot{p}_i + \sum_{i=1}^{n} \dot{p}_i^2 \leqq \rho^2 \sum_{i=1}^{n} F_i(p)^2 + 2\rho \sum_{i=1}^{n} p_i F_i(p) + \sum_{i=1}^{n} p_i^2 .$$

In view of the relation (b) $\sum_{i=1}^{n} p_i E_i(p) = -F_0(p)$, we have

$$(6) \qquad \sum_{i=1}^{n} p_i \dot{p}_i \leqq \frac{1}{2}\left(\rho^2 \sum_{i=1}^{n} F_i(p)^2 - \sum_{i=1}^{n} \dot{p}_i^2\right) - \rho F_0(p)$$

$$\leqq \frac{\rho^2}{2} \sum_{i=1}^{n} F_i(p)^2 - \rho F_0(p) .$$

On the other hand, also by definition,

$$\theta_i(p) \geqq \rho F_i(p) + p_i ,$$

which gives, in view of (1),

$$\dot{p}_i = \theta_i(p) - p_i \geqq \rho F_i(p) ,$$

and hence

$$(7) \qquad \sum_{i=1}^{n} \hat{p}_i \dot{p}_i \geqq \rho \sum_{i=1}^{n} \hat{p}_i F_i(p) .$$

Using (6) and (7),

$$\frac{dV}{dt} = \sum_{i=1}^{n} (p_i - \hat{p}_i)\dot{p}_i = \sum_{i=1}^{n} p_i \dot{p}_i - \sum_{i=1}^{n} \hat{p}_i \dot{p}_i$$

$$\leqq \frac{\rho^2}{2} \sum_{i=1}^{n} F_i(p)^2 - \rho[F_0(p) + \sum_{i=1}^{n} \hat{p}_i F_i(p)]$$

$$= \frac{\rho^2}{2} \sum_{i=1}^{n} F_i(p)^2 - \rho f(p) .$$

If now $p(t) \notin U(\hat{p}, \varepsilon)$, by Lemma 4, ($\gamma$) and (5),

$$\frac{dV}{dt} \leqq \frac{\rho^2}{2} \sum_{i=1}^{n} F_i(p)^2 - \rho f(p)$$

$$\leqq \frac{\rho^2}{2} L - \rho M = \frac{\rho L}{2}(\rho - 2M/L) = -\delta ,$$

where $\delta = (\rho L/2)(2M/L - \rho) > 0$.

On the basis of the above results (I) and (II), we always have

$$(8) \qquad \frac{dV}{dt} \leqq -\min[\delta, \rho f(p)] ,$$

from which it can be deduced that

$$\lim_{t \to \infty} V = 0 \qquad \text{(monotonically)} .$$

In fact, since $f(p) \geqq 0$ at all times, (8) implies that the non-negative function V of time is non-increasing. Accordingly, V monotonically converges to a non-negative limit η. If η were positive, $p(t) \notin U(\hat{p}, \eta/2)$ over time. In such a case, we can show, in exactly the

same fashion as in the proof for Lemma 4, that $f(p) \geqq \beta$ over time for a positive constant β. Therefore, in view of (8), dV/dt is majorized by a negative number, which yields a contradiction $\lim_{t\to\infty} V = -\infty$. This means that $\eta = 0$, and the proof is now complete.

5. RELATIONS TO THE ARROW-BLOCK-HURWICZ RESULT

5.1. Arrow, Block, and Hurwicz [3] recently proved the global stability of the competitive market for a special case, namely, for the case in which all goods are gross substitutes. One important feature of their result is the broad generality of price adjustment; the rate of price change is not necessarily proportional to excess demand but is assumed to be only a sign-preserving function of excess demand. In this connection one may point out that the right-hand side of equation (1) is of the same sign as $F_i(p)$ for $p > 0$.[9] Accordingly, if one assumes gross substitutability, and if the initial position p^0 is positive, the stability proved in this note for equation (1) is implied by the Arrow-Block-Hurwicz result. However, generally speaking, the results of this note are different in several points from theirs, as will be elucidated below.

5.2. First, it is noted that we have proved the existence of a tâtonnement path, namely, a solution of the differential equation discussed in Section 3 without assuming any behavioral conditions on excess demand functions like gross substitutability or the weak axiom of revealed preference, while Arrow, Block, and Hurwicz assume a special condition[10] on the shape of excess demand functions to achieve the corresponding objective. Thus, our mathematical derivation of Proposition 1 differs in nature from theirs, in that the exsitence of a tâtonnement path is discussed under general conditions, separately from the stability problem. Second, attention should be called to the somewhat different role played by the weak axiom of revealed preference here in comparison with the Arrow-Block-Hurwicz article. They proved global stability for a narrower type of adjustment process in which the rates of price change are proportional to excess demands, by deriving the weak axiom of revealed preference from gross substitutability. When, however, general sign-preserving cases were discussed, they were based directly on gross substitutability, and did not rely upon the revealed preference axiom implied by substitutability. In this note,

[9] It should also be noted that the right-hand side of (1) may not be of the same sign as excess demand on the whole domain of all non-negative $p \geqq 0$. In fact, if $p_i = 0$, and $F_i(p) < 0$, then $\theta_i(p) - p_i = 0$, so that it does not have a sign-preserving nature in the exact sense.

[10] See Lemma 6 in [3].

it has been shown that this axiom already suffices to establish global stability for a special, but sign-preserving, case. Finally, the initial position of the price tâtonnement in our case is assumed only to be non-negative and need not be strictly positive.

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REFERENCES

[1] ARROW, K. J. AND G. DEBREU, "Existence of an Equilibrium for a Competitive Economy," *Econometrica*, Vol. 22, No. 3 (1954).

[2] ARROW, K. J. AND L. HURWICZ, "On the Stability of the Competitive Equilibrium, I," *Econometrica*, Vol. 26, No. 4 (1958).

[3] ARROW, K. J., H. D. BLOCK, AND L. HURWICZ, "On the Stability of the Competitive Equilibrium, II," *Econometrica*, Vol. 27, No. 1 (1959).

[4] CODDINGTON, E. A. AND N. LEVINSON, *Theory of Ordinary Differential Equations* (New York: McGraw-Hill, 1955).

[5] GALE, D., "The Law of Supply and Demand," *Mathematica Scandinavica*, Vol. 3 (1955).

[6] HAHN, F., "Gross Substitutes and the Dynamic Stability of General Equilibrium," *Econometrica*, Vol. 26, No. 1 (1958).

[7] MCKENZIE, W. L., "On Equilibrium in Graham's Model of World Trade and Other Competitive Systems," *Econometrica*, Vol. 22, No. 2 (1954).

[8] ———, "On the Existence of General Equilibrium for a Competitive Market," *Econometrica*, Vol. 27, No. 1 (1959).

[9] ———, "A Contribution to the Theory of the Stability of Competitive Equilibrium," Report presented at *the Chicago Meeting of the Econometric Society*, December, 1958.

[10] NEGISHI, T., "A Note on the Stability of an Economy Where All Goods are Gross Substitutes," *Econometrica*, Vol. 26, No. 3 (1958).

[11] NIKAIDÔ, H., "On the Classical Multilateral Exchange Problem," *Metroeconomica*, Vol. VIII, No. 2 (1956); A Supplementary Note to [11] *Ibid.*, Vol. IX, No. 3 (1957).

[12] ———, "Stability of Equilibrium by the Brown - von Neumann Differential Equation," *Econometrica*, Vol. 27, No. 4 (1959).

[13] UZAWA, H., "A Note on the Stability of Equilibrium," Technical Report No. 44, Dept. of Economics, Stanford University (1957).

[14] ———, "Iterative Methods for Concave Programming," in K. J. Arrow, L. Hurwicz, and H. Uzawa, *Studies in Linear and Nonlinear Programming*, Stanford University Press, (1958).

[15] ———, "Walras' Tâtonnement in the Theory of Exchange," *Review of Economic Studies*, forthcoming.

[16] WALD, A., "Über die Produktionsgleichungen der Ökonomischen Wertlehre," *Ergebnisse eines mathematischen Kolloquiums*, Heft 7 (Vienna, 1936).

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AN ESTIMATE OF NATIONAL INCOME OF CZECHOSLOVAKIA, 1946–1953*

By Boris P. Pesek

During the past few decades the national income accounts have been used increasingly as a tool for the analysis of the structure and growth of the national economy. In most Western countries these accounts are prepared either officially or semi-officially; for the countries of the Soviet bloc there are some official data of questionable value and many careful studies by Western scholars dealing with the U.S.S.R. There is, so far as I have been able to discover, nothing in this field which deals with other countries belonging to this Soviet bloc of nations, despite the fact that their economic potential is far from negligible, and their economic development perhaps more interesting than that of the U.S.S.R.

At the beginning of Soviet rule, Russia was an economically underdeveloped, if not an undeveloped, semi-feudal country, with characteristics which made it unsuitable for a communistic experiment. As a result, the evaluation of the subsequent performance of Soviet economy has led to a bi-polarization of views. On the one hand is found the view that economic organizations of the same type are bound to suffer from the same ills, regardless of whether they are run by Frenchmen, Russians, or Americans, and that therefore there is nothing atypical in the Russian performance. On the other hand, there is the belief that it is a disaster for the idea of planning that Russia should have been the country where it was first tried out.

Czechoslovakia, in contrast, was at the time of transition an advanced industrial nation. To measure the economic growth of this nation after the imposition of the Soviet-type planning appears to be an interesting undertaking. I have undertaken this task while working on a somewhat different problem.[1] A highly digested description of the techniques I used to arrive at the national income estimates covers well over one hundred pages. I fear that such a manuscript would strain the patience of most readers. For that reason I intend to acquaint the reader with only the bare results of my work. The

* Manuscript received June 1, 1959.

[1] Boris P. Pesek, Monetary Policy of Czechoslovakia, 1945-53 (unpublished Ph. D. dissertation, The University of Chicago, 1956).

period covered by my national income accounts starts at the end of the war and ends with the year of the monetary reform of 1953. In that and the following year Czechoslovakia undertook some basic organizational changes which made it impossible for me to utilize, for the subsequent period, the national income framework suitable for the 1946–53 period.

1. SOME BASIC CHARACTERISTICS

Czechoslovakia, with which I deal in this article, was at the time of transition an "advanced industrial" or "incipient decline" type of community.[2] In 1945 the country was well supplied with productive equipment and managerial and administrative skills. Besides this original endowment with both natural and created resources, the population of Czechoslovakia was favorably inclined toward the elimination of private enterprise and the introduction of centralized economic planning; this is an important quality of the human resource without which the planned economy is bound to encounter great difficulties. I cannot here try to analyze the reasons for this favorable attitude. Rationalism in the realm of popular philosophy, coupled with an almost complete lack of education in economic science, the thorough-going cartelization of the pre-war economy, the bitter experience with the Great Depression, and the over-full employment introduced immediately afterward by the German planning machinery all contributed to the creation of this attitude. As a result, at least one-half the population was not only able, but relatively willing, to embark on an experiment in planning.[3]

The *coup d'état* of February 1948 had, from an economic standpoint, two mutually contradictory, and therefore partly neutralizing, effects on this quality of the human resource. On the one hand, the majority of the population did not approve of a seizure of power by one party alone;[4] this meant that the planners lost some of the willingness of the people to cooperate. On the other hand, all criticism pointing

[2] This classification was devised by Theodore W. Schultz in *The Economic Organization of Agriculture* (New York: McGraw-Hill, 1953), p. 32. It covers communities where 25 per cent or less of the income of the consumer is spent on farm services involved in food production.

[3] In the elections of 1946 two parties pledged to vigorous centralized planning received an absolute majority of votes: the Communists 38 per cent and the Social Democrats 13. In addition, a third party (National Socialists), pledged to a moderate degree of over-all planning received an additional 30 per cent of all the votes.

[4] Towards the end of 1947 public opinion polls indicated serious Communist losses. Rather than wait for the elections scheduled for May 1948, the Communist party seized power towards the end of February 1948. Cf. Busek et al., *Czechoslovakia*, (New York: Praeger, 1957).

out the errors and mistakes of the planners was removed, and all intra-governmental controversies about the goals of the economic plan ceased.[5].

Whatever the strength of these forces, it is certain that the planners in Czechoslovakia gained control of a much more favorable collection of physical and human resources than they ever succeeded in acquiring elsewhere. This is true even if East Germany is taken into account. While there is some superficial similarity between the material endowment of Czechoslovakia and East Germany, in the latter country war destroyed many resources, and reparation payments removed many more. The human resources, though perhaps of equal quality at the start, were much less suitable for the planning experiment because of the fact that the entire governing machinery rested on the shoulders of victorious enemies.

The post-war economic developments in Czechoslovakia can be divided into three distinct periods. During the years 1945–46, the economy was being prepared for centralized economic planning. Industrial enterprises with more than 250–500 employees were nationalized together with the banking system; planning machinery was set up, and the economic system was stimulated to convert from war production to peace-time production. In 1947–48, the planners were permitted to carry out a test run called the Two-Year Plan. It revealed many shortcomings in the planning machinery and a lack of control over some important segments of the national economy. The planners incurred indirect, but substantial, criticism from the non-Marxian governing parties. In February 1948, the *coup d'état* transferred all power to the hands of the Communist Party. All enterprises with more than 50 employees were nationalized, and economic pressure was exerted on smaller enterprises, many of which were forced to liquidate or become members of state-ruled cooperatives. The new Consititution and a Planning Act placed practically unlimited powers in the hands of the Government and the State Planning Office. In the fall of 1948, the Government announced a new Five-Year Plan for the period 1949–53. This Plan was revised upwards in 1950 and 1951, and twice downward in 1953.

[5] This is not meant to deny constructive value to criticism in general. I claim merely that planning, like democracy, can successfully survive only if the discussants agree on fundamentals and pursue only incrementally different policies. Cf. Kenneth J. Arrow, *Social Choice and Individual Values* (New York: John Wiley & Sons, 1951), pp. 59 ff and R. A. Dahl and C. E. Lindblom, *Politics, Economics, and Welfare* (New York: Harper and Brothers, 1953), pp. 294 ff.

2. SOURCES

The official national income data are of limited use on two counts. First of all, I found it difficult to believe that these data express reality. For instance, they state that the national income rose by more than 70 per cent from 1948 to 1953 (Table 1). If we take into consideration the facts that the year 1948 was a year of full employment and that the war-caused disorganization was largely overcome by that time, such a rise would be unprecedented and must be rejected unless a great deal of additional supporting evidence is available, as it is not.

The second reason for rejection is the fact that only one final figure is offered for each year: the total national income. Even this figure is offered (see Table 1) in two inconsistent versions. As long as the individual components of this complex figure are not known; as long as even the deflating method is not known, there is no way by which the official data can be understood, analyzed, and checked against other available evidence. They have to be either accepted or rejected on the strength of faith alone.

For my own accounts I used a great variety of sources, starting with the lengthy annual reports describing the fulfillment of the plan and ending with isolated press reports about the price of this or that commodity. In a number of cases I had to select one of several available figures measuring apparently the same thing. In other cases, crude estimates were all that could be accomplished.

TABLE 1

TWO OFFICIAL ESTIMATES OF NATIONAL INCOME OF CZECHOSLOVAKIA, 1946-53

Year	National Income[1] (1937 crowns)	Index of National Income[2]
1937	56.5	100
1946	47.4	n.a.
1947	51.3	n.a.
1948	56.5	97
1949	63.6	110
1950	68.2	122
1951	78.1	133
1952	94.0	145
1953	n.a.	154

Sources: (1) *Czechoslovak Economic Bulletin*, Jan. 1, 1952.
Plánované Hospodářství, No. 9, 1951, p. 671.
(2) *Statistická Ročenka 1957* (Prague: Orbis, 1957), p. 291.

3. PERSONAL INCOME AND DISPOSITION OF INCOME

For my estimates I used the framework constructed for the national income accounts by the United States Department of Commerce.[6] I have found it convenient, however, to divide the personal income accounts into four segments, each of which has several subdivisions. These segments cover incomes of self-employed persons, incomes of employed persons, incomes derived from rents and interests, and incomes derived from transfer payments.

Due to the character of the Czechoslovak economy, incomes of employed persons account for an exceptionally large fraction of personal income. Repudiation of money assets in 1945 and nationalization left many persons without means of support; this, together with the establishment of a welfare state, caused transfer payments to assume great importance. Wages, salaries, and transfer payments account in 1946 for 79 per cent, and in 1953 for 87 per cent, of personal income.

For both these segments, comparatively adequate statistical data could be obtained. Transfer payments were obtainable directly from the state budget. For wages and salaries, I could gather per worker averages. After constructing statistical series on the occupational distribution of the labor force, I used the average figures to calculate total wage and salary bill. Because of the importance of this bill in the total personal income, a check of the resulting figures was especially desirable. Independent evidence permitted such check of my estimates for the year 1946 and for the period 1946–50. Both tests bear favorably on the accuracy of the estimates.

Rents and interest income represent a negligible part of personal income. This is due to the fact that both rents and the interest rate became fixed by an official decree in 1939 and did not change thereafter. In addition, the blocking of money assets in 1945 removed a large part of the base from which interest income was derived.

Incomes of self-employed persons were the most difficult to estimate. No really satisfactory evidence was obtainable. In my estimate, these incomes account for 21 per cent of personal incomes in 1946 and for 12 per cent eight years later. In view of the general characteristics of the economy these percentages appear to me realistic, and I have some confidence that even the crude estimating techniques utilized for this segment of personal incomes are satisfactory.

Closely related to personal income should be the personal consumption expenditures. For most years data were available from official sources. One of the most important tests of accuracy of these data

[6] United States Department of Commerce, *National Income, 1954 Edition* (Washington D. C. : U. S. Government Printing Office, 1954).

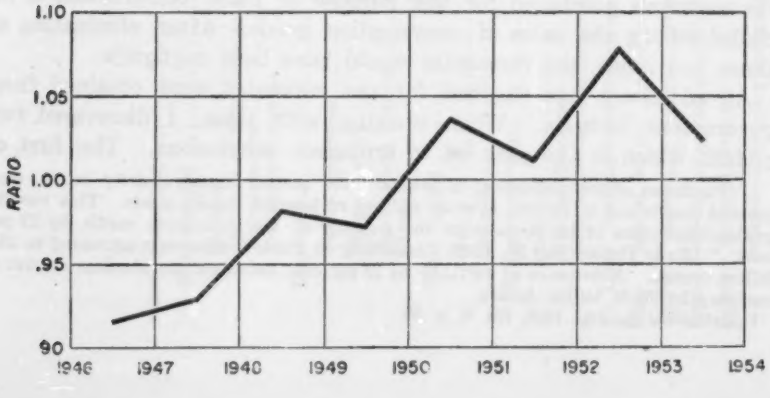
consisted in the deflation of the consumption series. In money terms consumption expenditures almost quadrupled from 1946 to 1953. If the data on consumption expenditures and the price index for consumption goods are dependable, deflation on *a priori* grounds should show only slow movements in real per capita consumption over time. Thus deflation provides a simultaneous check on both the consumption figures and the price index used. From Table 4 it can be seen that the data passed this test satisfactorily. Personal disposable income is connected with personal consumption expenditures through personal savings. Because the measurements of both magnitudes originate from completely different sources and because these magnitudes are, at the same time, economically related, a comparison of them may be used for the evaluation of the dependability of both estimates. In view of the importance of this comparison, I have plotted in Figure 1 the ratio of personal consumption expenditures to personal disposable income.

(a) While equality between these two series is neither required nor desired, wide discrepancies between them would make the accuracy of both series suspect. We observe that throughout this period both series are close to each other.

(b) Savings shift from a higher plateau for the years 1946–47 to a lower plateau in the years 1950–53. This is what we would expect in view of the fact that the radical nationalization ordered in the Spring of 1948 closed investment outlets and that prices after a period of stability started to rise very sharply in the last quarter of 1948 and continued to rise for the rest of the period. In addition, social security provisions improved substantially.

FIGURE 1

RATIO OF PERSONAL CONSUMPTION EXPENDITURES TO PERSONAL DISPOSABLE INCOME, 1946-53



(c) There are two peaks in the dissavings during the latter part of the period, existence of both of which are confirmed by independent evidence. In 1950, there occurred the Polish monetary reform, and the Korean War started. Extraordinary purchases were commented upon by the Government and blamed on the apprehension caused by the Polish reform; I believe that the war scare was at least of equal importance. In 1952, wide-spread rumors of an impending monetary reform caused dissavings again. A somewhat cryptic governmental estimate of the size of the dissavings is remarkably close to my own.[7]

On all these grounds I feel justified in my belief that Personal Income and Disposition of Income Estimates presented in Table 2, are a good expression of reality.

4. GROSS NATIONAL PRODUCT

Next I calculated the Gross National Product. Again I used the framework of the U.S. Department of Commerce. However, I simplified this framework considerably to make it useful for my purpose. Disposition of Gross National Product contains disposable income (equal to consumption plus private savings), enterprise income, statistical discrepancy including residual, and tax receipts of the government. Sources of Gross National Product contain personal consumption expenditures, gross domestic investment, net foreign investment, and government expenditures on goods and services (excluding governmental investment expenditures). Personal investment expenditures are not included in my totals for two reasons: (a) The market value of these investments amounted to 6.2 billion crowns in 1948, 6.0 in 1949, 5.0 in 1950, 4.6 in 1951, 4.6 in 1952, and 4.8 in 1953.[8] However, all these investments were built by the owner and the market value includes imputed wages. (b) A substantial amount of the materials purchased for the purpose of home construction is included among the sales of consumption goods. After elimination of these two items, the remainder would have been negligible.

All additional data required for the estimates were obtained from government budgets. When working with them, I discovered two pitfalls which in the past led to erroneous conclusions. The first of

[7] "Purchases of the population, in several short periods extraordinarily heavy, were covered mostly out of current incomes and out of hoarded money assets. This was the predominant cause of the decrease of the quantity of the circulating media by 23 per cent..." (*Rudé Právo*, July 24, 1952). According to Table 2, dissavings amounted to 25.6 billion crowns. A decrease of currency by 23 per cent decreased the absolute amount of currency by 26-31 billion dollars.

[8] *Statistické zprávy*, 1958, No. 6, p. 20.

TABLE 2

CZECHOSLOVAKIA

PERSONAL INCOME AND DISPOSITION OF INCOME, 1946–53[a]

(billions of crowns)

	1946	1947	1948	1949	1950	1951	1952	1953
PERSONAL INCOME	149.323	176.070	199.052	230.998	285.001	312.667	345.675	449.829
Self-employed	31.405	33.762	32.365	30.432	33.718	35.194	35.689	53.785
Farmers	14.750	13.830	15.352	17.593	21.573	24.740	22.156	32.313
Members of farm collectives	—	—	—	.675	1.388	1.591	5.872	15.405
Entrepreneurs	15.346	18.415	15.397	12.164	10.757	8.863	7.661	6.607
Professional persons	1.309	1.517	1.616	—	—	—	—	—
Employed	106.616	127.338	147.347	164.403	205.263	228.633	254.668	314.610
Salary employees	30.728	36.589	41.661	47.818	61.848	62.055	72.782	95.012
Wage workers	68.359	83.223	97.122	107.857	134.024	155.164	172.341	210.407
Agricultural workers	6.690	6.414	7.197	7.151	7.372	9.016	6.827	5.819
Others	.839	1.112	1.367	1.576	2.019	2.398	2.718	3.372
Other income	3.672	3.955	4.127	4.234	4.234	4.234	4.234	4.234
Rents	3.512	3.512	3.512	3.512	3.512	3.512	3.512	3.512
Interests	.160	.443	.615	.722	.722	.722	.722	.722
Transfer payments	12.930	16.975	21.513	31.929	41.796	44.606	51.084	77.200
Pensions	6.045	7.136	9.913	15.888	22.004	24.764	25.663	
Family allowances	2.466	3.596	3.858	3.945	4.973	6.000	6.721	77.200
Cash sickness compensations	4.419	6.243	7.742	12.096	14.819	13.842	18.700	
Less: Personal contributions for social security	5.300	5.950	6.300	—	—	—	—	—
Less: Personal tax and non-tax payments	9.801	11.604	13.125	15.095	23.100	9.300	20.100	25.817
Equals: DISPOSABLE PERSONAL INCOME	139.522	164.466	185.927	215.903	261.911	303.367	325.575	424.012
Less: Personal consumption expenditures	127.883	152.362	182.565	219.958	271.428	306.825	351.193	433.947
Equals: Personal savings	11.639	12.104	3.362	−4.055	−9.517	−3.458	−25.618	−9.929

[a] Because of rounding components may not add up to totals.

TABLE 3

CZECHOSLOVAKIA

DISPOSITION AND SOURCES OF GROSS NATIONAL PRODUCT, 1946-53[a]

(billions of crowns)

		1946	1947	1948	1949	1950	1951	1952	1953
1.	Consumer purchases	127.883	152.362	182.565	219.958	271.428	306.825	351.193	433.947
2.	Gross savings								
3.	Gross personal savings	11.639	12.104	3.362	−4.055	−9.517	−3.458	−25.618	−9.929
4.	Gross enterprise savings[b]								
5.	Statistical discrepancy[b]	1.219	10.147	26.260	34.364	57.747	20.614	26.768	−39.539
6.	Taxes (including social security contributions minus transfer payments)	39.687	42.095	61.069	108.887	140.070	214.682	287.306	311.439
7.	DISPOSITION OF GROSS NATIONAL EXPENDITURES	180.428	216.708	273.256	359.154	459.728	538.663	639.649	695.922
8.	Consumer purchases	127.883	152.362	182.565	219.958	271.428	306.825	351.193	433.947
9.	Gross domestic investment	16.250	24.750	43.736	89.041	122.735	147.316	157.830	167.000
10.	Investment by enterprise	5.000	16.239	24.422	29.400	43.800	22.800	31.000	37.000
11.	Investment by government	11.250	8.511	19.314	59.641	78.935	124.516	126.830	130.000
12.	Government investment (+) or disinvestment (−) in emergency stocks of consumer goods	0	0	0	0	0	1.600	18.196	−22.213
13.	Net foreign investment	−8.600	−7.200	−4.300	.909	4.430	−6.250	.148	5.942
14.	Government purchases of non-capital goods and services	44.895	46.796	51.255	49.246	61.135	89.172	112.282	111.246
15.	SOURCES OF GROSS NATIONAL PRODUCT	180.428	216.708	273.256	359.154	459.728	538.663	639.649	695.922

[a] Because of rounding components may not add up to total.

[b] Items 4 plus 5 are a residual.

these is due to the fact that appropriation acts have almost no relationship to actual cash performance; while figures on the former are readily obtainable, the latter could be got only with great difficulty. It appears that cash performance departs from the appropriated sums, on the average, by 20 per cent per year; if this discrepancy is neglected, it becomes a potent source of error. The second difficulty lies in the fact that in the 1946–53 period the government twice changed drastically the content of the budget.[9]

Table 3 presents the estimates of the Gross National Product of Czechoslovakia during the period 1946–53.[10]

5. GROSS NATIONAL PRODUCT IN REAL TERMS

It is of general interest, and it was essential for my purpose, to know what part of the changes in the Gross National Product are due to the growth of factors of production used in the economy and what part of them only to the changes in the prices of those factors.[11]

The method I used to arrive at the Gross National Product in real terms had to be different for each segment of my estimates. Some of them were deflated by a price index, some of them were estimated both in money and real terms with implicit price deflators resulting as a by-product, and one remaining segment was deflated by an index based partly on implicit price deflators, partly on a price index. Obviously, the availability of statistical data dictated the choice.

For consumption goods, I calculated a price index of my own. It is based on a revised version of the official cost of living index. The official index was published until the end of 1948. I recalculated this index for the period 1946–48 to eliminate certain items which were erroneously included in it (direct taxes). Then I extended the index to cover the period 1949–53. In this way I obtained an index of prices of rationed commodities. Using the same basket of commodities, I calculated a price index expressing the changes in prices of these commodities when sold on the black market in the period 1946–48

[9] The reader acquainted with published figures on Czechoslovak government expenditures and government investment expenditures will notice that my own figures differ sharply from them. This is due to the fact that in all other publications known to me, including those of the United Nations, the first pitfall is overlooked and the second pitfall only mentioned, and adjustment for it is not attempted.

[10] On May 31, 1953 all magnitudes expressed in terms of money were divided by five and monetary units themselves exchanged in the ratio 1:5. This measure has only accounting significance. To facilitate easy comparison of all figures I consistently used the pre-reform accounting unit.

[11] I shall argue later that my accounts measure resources used up rather than output produced. Therefore I speak here about quantities and prices of factors of production.

TABLE 4

INDEX OF PERSONAL CONSUMPTION EXPENDITURES IN TERMS OF CURRENT CROWNS AND IN TERMS OF 1947 CROWNS, CZECHOSLOVAKIA, 1946-53

Year	Index of consumption in current crowns	Index of per capita consumption in 1947 crowns
1946	83.9	79.4
1947	100.0	100.0
1948	119.8	97.3
1949	144.4	88.0
1950	178.2	103.5
1951	201.4	104.5
1952	230.5	101.1
1953	284.4	113.6

and the changes in the prices of these commodities when sold on the officially sponsored "free market" in the period 1949–53.

Good fortune provided me with some data on the distribution of the purchases of the population between free and rationed commodities. I used this information to prepare a unified price index covering both sets of prices. The final index of prices of consumption goods, when used to deflate consumption expenditures expressed in money terms, transformed a sharply rising series to a series largely stable over time (Table 4).

The real consumption series behaves in a very consistent fashion. It rises in the wake of the post-war consolidation, drops slightly in 1948 as foreign help ceases, and then grows slowly. The apparent interruption of this growth in 1951 and 1952 and the sharp rise in 1953 is really misleading. In the two former years the Government started to acquire large emergency inventories of consumer goods, which were placed on the market at the start of the monetary reform of 1953. The only year which appears suspicious is the year 1949. I find it hard to believe that real consumption in that year actually dropped as much as Table 4 indicates. It may be that there is some error in my price index for that year though careful recheck failed to discover any.

Figures on gross domestic investment became available both in money and in real terms; therefore these two magnitudes could be used to calculate implicit price deflators for investment expenditures. Government expenditures were divided into three segments:

(a) Investment expenditures were available both in money and real terms and therefore yielded directly an appropriate implicit deflator.

TABLE 5

INDICES OF PRICES FOR MAJOR SEGMENTS OF THE GROSS NATIONAL PRODUCT, CZECHOSLOVAKIA, 1946-53 (1947 = 100)

Item	1946	1947	1948	1949	1950	1951	1952	1953
Consumption goods, emergency stocks	106.5	100.0	122.3	162.3	169.1	187.6	219.7	238.7
Investment goods	92.0	100.0	122.2	215.0	211.8	254.2	241.6	266.1
Government goods and services	88.1	100.0	113.5	154.1	160.7	185.1	187.0	203.9
GROSS NATIONAL PRODUCT	99.5	100.0	121.1	171.2	176.2	203.8	217.6	235.3

(b) Investment expenditures in emergency stocks of consumption goods were deflated by the index of prices of consumption goods.

(c) Ordinary expenditures of the government were deflated by a crude index which combines index of prices of investment goods and index of prices of wages and salaries of government employees.

Net foreign investment were left undeflated because of lack of data. The error due to this failure to deflate must be small because net foreign investment is such a negligible part of the Gross National Product.

In Table 5, I present the price indices applicable to the individual components of the Gross National Product.

I take it to be a favorable sign that these indices remain in a reasonable proximity to each other throughout this period. The index showing the greatest rise, the index for investment goods, is higher than the index for consumption goods by, at most, 36 per cent; the first of these is based on implicit deflators while the second one is an ordinary price index. For the sake of comparison I mention that in the United States the widest divergence between prices in recent times occurred in 1943, when the index for government purchased goods was 31 per cent higher than the index for private investment expenditures at the other end of the price spectrum.

In Table 6, I present the estimate of Gross National Product in real terms and several indices based on this estimate. Again, the year to year movements seem to be consistent with other information about the Czech economy. The year 1949 was the year of radical organizational and ownership changes caused by the start of the first Five-Year-Plan. Production recovered next year and continued to increase until 1953, when serious inflation, confiscation of 71-87 per cent of private money assets, and resulting disturbances of peace once again interrupted the upward climb of output.

TABLE 6

CZECHOSLOVAKIA

GROSS NATIONAL PRODUCT IN CONSTANT CROWNS, 1946-53

(billions of 1947 crowns)

	1946	1947	1948	1949	1950	1951	1952	1953
Consumer purchases	121.308	152.362	149.166	135.567	160.513	163.533	159.924	181.796
Gross business and public domestic investment	17.663	24.750	35.790	41.414	57.949	57.953	65.327	62.758
Government investment (+) or disinvestment (−) in emergency stocks of consumer goods	0	0	0	0	0	.853	8.453	−9.306
Net foreign investment	−8.600	−7.200	−4.300	.909	4.430	−6.250	.148	5.942
Government purchases of non-capital goods and services	50.959	46.796	45.070	31.957	38.043	48.175	60.044	54.559
GROSS NATIONAL PRODUCT	181.300	216.708	225.726	209.847	260.935	264.282	295.896	295.749
INDICES:								
Gross National Product per capita	84.9	100.0	103.5	96.2	118.5	118.6	131.3	129.8
Gross National Product per worker[a]	n.a.	100.0	103.9	96.6	119.5	120.7	134.9	132.9
Gross National Product per worker[b]	84.6	100.0	103.0	94.5	114.5	114.4	125.8	120.9

Notes: (a) Official labor force data.

(b) Adjusted labor force data. The main adjustment occurs in the calculation of farm population. Official data count as employed all those living on farms who are older than 14 years. I have adjusted for housewives, students, and part-time industrial workers

6. COMPARISON WITH OFFICIAL FIGURES

It would be desirable at this point to compare my estimates of the national income with the official estimates. This is, however, an exceptionally difficult task. It is impossible to compare the accounts in absolute terms because the official data consist only of indices. Only a comparison of relative changes over time can be carried out. Even this procedure is not very satisfactory because the government figures are left undefined. The entire description of the composition of the government accounts, of the techniques of measurement, and of the technique of deflation into monetary units of constant purchasing power reads:

> National income is a part of the social product left over after replacement of the means of production used up during each year. Social product is the sum of all the physical goods—investment and consumption goods—produced during the year. National income, embodying newly expended labor, contains the produced consumption goods and the part of investment goods destined for increased production.[13]

Nevertheless, it may be useful to present the official figures side by side with my estimates (Table 7).

According to the official index, Czechoslovak national income grew from 1948 to 1952 by 10 per cent per year. This regular, steady growth is hard to believe when we take into account the economic history of that period. The year 1949 was the first year after the

TABLE 7

COMPARISON OF CZECHOSLOVAK NATIONAL INCOME DATA

	Official Index of National Income	Pesek Index of Gross National Product	Annual Percentage Changes	
			Official Index	Pesek Index
1946	n.a.	80.3		
1947	n.a.	96.0	—	—
1948	100	100.0	—	—
1949	110	92.9	+10.0	−7.0
1950	121	115.6	+10.0	+24.4
1951	133	117.1	+10.0	+1.7
1952	146	131.1	+9.7	+11.9
1953	155	131.0	+6.1	−0.1

Sources: Official index: *Statistická Ročenka*, 1958, p. 29.

[13] *Statistická Ročenka*, 1958, p. 17.

Communist seizure of power; the remaining private sectors of industry were nationalized, and the Five-Year Plan was instituted. Under these circumstances it is not surprising that qualitative evidence suggests a substantial degree of confusion and disorganization. The shift of emphasis away from consumption goods toward heavy investment goods only compounded these difficulties. Subsequently, the Five-Year Plan was revised upward in 1950 and 1951, twice downward in 1953. To credit the official growth figures would thus necessitate a conclusion that the original Five-Year Plan provided for a very uneven pattern of economic growth.

Least credible of all is the last year of the period covered. According to innumerable official complaints the dual-price system was destroying all incentives to work. Absenteeism in industry rose from 6.8 per cent in 1947 to 21 per cent in 1952[13] and was still rising in the first half of 1953. In extreme cases it reached the level of 48 per cent.[14] Finally, in May, the government abolished the dual-price system, repudiated 70–87 per cent of privately held money assets and 100 per cent of state and local debt.[15] Isolated riots of industrial workers followed. Nevertheless, the official data claim that output continued to grow, though at a somewhat lower rate.

My estimates of gross national product seem to reflect much better the institutional changes and upheavals which occurred during this period. Output declined during the nationalization and reorganization year 1949 but recovered and forged ahead in the following year. In 1951 the impetus given to the economy by improved organization was spent. In 1952 the introduction of incentive pay and the tightening of financial discipline led to another jump. Finally, in the first half of 1953 the stresses introduced by the investment drive proved to be unbearable, and for all practical purposes the system broke down late in the Spring. After the reform, the supply of labor increased as workers attempted to restore their depleted money assets to the desired level; output increased as a consequence. My own estimates suggest that the losses during the first half of the year were almost made up during the second half. Thus, on the whole, my estimates seem to reflect better the history of that period.

There is, suprisingly enough, extreme similarity between my consumption series and the official one (Table 8), despite the fact that

[13] *Rudé Právo*, Jan. 23, 1953.

[14] Czechoslovak Home Service, Jan. 10, 1953, 1815 GMT.

[15] Cf. B. P. Pesek, "Monetary Reforms and Monetary Equilibrium," *The Journal of Political Economy*, Oct. 1958, p. 383.

TABLE 8

COMPARISON OF CZECHOSLOVAK CONSUMPTION ESTIMATES

Year	Official Index*	Pesek Index (output)	Pesek Index (purchases)	Percentage Changes	
				Official Index	Pesek Index (output)
1946	n.a.	81.3	81.3		
1947	n.a.	102.1	102.1	—	—
1948	100	100.0	100.0	—	—
1949	103	90.9	90.9	+3.0	−9.1
1950	120	107.6	107.6	+16.5	+18.4
1951	122	110.3	109.6	+1.6	+2.5
1952	124	112.8	107.2	+1.6	+2.2
1953	120	108.9	121.9	−3.3	−3.5

* The official index is probably an output index. In my index "purchases" plus purchases (minus sales) of emergency stocks of consumer goods equal "output".
Source of official index: *Statistická Ročenka*, 1958, p. 29.

my series is based on consumption expenditures deflated by my price index, while the official series seems to be based on a quantity index.

The only real difference comes about in the year 1949: the official figures show a three per cent increase while my estimates show a nine percent drop. In the subsequent years the differences in the annual percentage changes are so small as to be negligible. I am convinced that the nationalization of (mainly) consumption industries in 1949 and the precipitate shift to the output of investment goods led to a decline in consumption. It is conceivable, however, that the actual decline was somewhat smaller than my estimates indicate.

The two investment indices are identical for the simple reason that I used the official figures. Some reservations as to the meaning of these investment figures shall be registered later on. My series dealing with the last sector, the government, cannot be compared because there are no official estimates of the size of the contribution of the government sector.

In summary I would say that my estimates appear to me to be more useful than the official estimates for the following reasons: (1) My estimates contain defined components, while in the case of the official estimates we have to guess the meaning of each series. (2) My estimates are expressed in monetary units, and thus all the components of my accounts are additive; the official estimates are expressed partly in index numbers so that the components of these accounts are non-additive. (3) My estimates are built up from a number of segments, each of

which was and can be separately analyzed and checked, while the official estimates contain only three series: consumption, gross investments, and national income [16] (4) The components entering my gross national product accounts are, on the whole, based on the official figures. Thus the official national income series are presumably in conflict with other related published series. (5) My accounts contain services excluded from the official index. Because services grew very slowly—and in some cases even declined—the official index necessarily overstates the actual rate of economic growth. (6) My estimates seem to be more consistent with our qualitative knowledge on developments in the Czechoslovak economy than are the official estimates.

7. CONCLUSION

The Gross National Product estimates presented here in both real and money terms are based on fragmentary evidence and must therefore be considered to be very tentative. The degree of confidence which can be placed in them depends on the use to which they are to be put.

As a measure of final useful goods produced by the Czechoslovak economy they contain, I believe, an upward bias, the size of which it is impossible to guess. National income accounting is ill-suited to measure this magnitude dependably in a situation like that of Czechoslovakia. One of the difficulties originates in the assumption that value added by the government sector is measured by the cost of governmental services. While technically this may be the best assumption, the rationalization behind it is somewhat dubious in democratic countries and completely false in countries in which democratic processes are not operative. It is based on a claim that were this equality not true, the electorate could always adjust the cost of governmental services appropriately. But in the case of Czechoslovakia it is hard to believe, for example, that the cost of the secret political police measures adequately the value added to the national product by these services.

The second difficulty leading to an overestimate of the Gross National Product is the valuation of government investment at cost. After the Czechoslovak Government entered the investment field on a large scale, a substantial part of investment funds, according to official admission, was wasted on hastily conceived and badly executed projects.[17]

[16] Cf. *Statistická Ročenka*, 1958, p. 94.

[17] Space limitations do not permit me to document this claim. This I have done in my chapter on "Economic Policies Since 1948" in J. Hajda, ed., *A Study of Contemporary Czechoslovakia* (New Haven: Human Relations Area Files, Inc., 1955).

The accounting problem which arises under these circumstances may be compared to a problem of how to value a trans-Atlantic steamer built on the shores of the Great Salt Lake in Utah. In both cases the existing techniques of national income accounting force us to add obviously wasted goods and services to the Gross National Product. In other words, we are measuring inputs which, because of increasing wastage, grow faster than useful outputs of investment goods.

Instead of measuring useful output of goods and services, in such circumstances we measure rather the amounts of goods and services used up in the economy, regardless whether they were put to "rational" or "irrational" uses.[18] The fact that I have decided to present these accounts to the reader demonstrates my belief that my accounts are reasonably good measure of the resources which the Czechoslovak Government mobilized for productive service.

Michigan State University, U.S.A.

[18] "Ration al use" in this context is understood to mean use approved either by the market mechanism or by the majority decision of the body politic.

BOOKS RECEIVED

T. E. Easterfield: *Productivity Measurement in Great Britain, A Survey of Recent Work*, London: Department of Scientific and Industrial Research, 1959, pp. 79.

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Council of Europe: *Statistical Data, 1958*, Strasbourg, 1959, pp. 399.

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